On the Capacity of Multi-user Two-way Linear Deterministic Channels

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***Abstract*—In multi-user two-way channels nodes are both sources and destinations of messages. This allows for "adaptation" at or "interaction" between the nodes – the next channel inputs may be a function of the past received signals at a particular node. How to best adapt is key to two-way communication problems, rendering them complex and challenging. However, examples exist of channels where adaptation is *not* beneficial from a capacity perspective; it is known that for the point-to-point two-way modulo 2 adder and Gaussian channels, adaptation does not increase capacity. Recently, it was shown that the two-way modulo-2 additive versions of the multiple-access / broadcast (MAC/BC respectively, in the two directions), the Z channel and the interference channel also have capacity regions equal to two parallel one-way versions of the channels. In this work we show that the same is true for the linear deterministic multi-user two-way channels which approximate their Gaussian counterparts at high SNR, which include the two-way MAC/BC channel, the two-way Z channel, and the two-way interference channel under some adaptation constraints. For all three channel models we obtain the capacity region, which is that of two one-way channels in each direction, which may be achieved without the use of adaptation.**

I. Introduction

In two-way communication, in which two users exchange messages over the same shared channel, nodes may be sources and destinations of messages. This permits them to adapt their channel inputs to their past received signals (which we loosely term "interaction" or "adaptation"), as first considered in the point-to-point two-way channel by Shannon [1]. Shannon's inner and outer bounds [1] are not tight in general, and have since been tightened, but a general computable formula for the capacity of the two-way channel remains open.

For point-to-point two-way channels, capacity is known for several channel models where the interaction between ones own signal and that of the other may be resolved. For example, in the two-way modulo 2 binary adder channel where $Y_1 = Y_2 = X_1 + X_2$ for binary X_1, X_2 and modulo 2 addition, the capacity region is one bit per user per channel use, as each user is able to "undo" the effect of the other, something that is not possible (at least not in one channel use) for the elusive binary multiplier channel with $Y_1 = Y_2 = X_1X_2$. In the binary modulo 2 adder channel, information independently flows in the $\rightarrow$ and the $\leftarrow$ "directions" and nodes need not interact to achieve capacity. In a similar fashion, the capacity of a two-way Gaussian point-to-point channel is equal to two parallel Gaussian channels, which may be achieved without the use of adaptation at the nodes [2]. "Adaptation" or "interaction" is said to take place when the next channel input of a node is a non-trivial function of that node's past received signals.

Contributions. In this paper, we are interested in determining whether similar statements can be made in multi-user channels rather than point-to-point channels. One expects this not to be true in general. For example, in multi-user Gaussian channels one may intuitively expect adaptation to allow for correlation between channel inputs which may translate to coherent gains. However, as we will see, there exist multi-user channels for which adaptation is useless.

We introduce three two-way multi-user linear deterministic channels useful for approximating Gaussian channels at high SNR [3]. Specifically, we consider the linear deterministic two-way a) Multiple Access / Broadcast channel (MAC/BC with 4 messages), b) Z channel (6 messages), and c) interference channel (IC, 4 messages). We ask whether adaptation may increase the capacity regions beyond that of two parallel one-way multi-user channels in the $\rightarrow$ and $\leftarrow$ directions. We will show that it does not for the first two channel models. For the two-way interference channel, we will show that partial adaptation where only 2 of the four nodes may adapt, can "block" the two-way information flow and destroy the ability to relay / cooperate. Under this constraint, we obtain the capacity region which is equal to two non-adaptive interference channels. In deriving outer bounds for these channel models, we use carefully chosen genies and Markov chain structures.

Related Work. The first of our three channel models is a linear deterministic MAC/BC channel. The capacity regions of the linear-deterministic one-way MAC and BC channels were obtained in [4]. An achievable rate region and an outer bound of a similar multi-user (multiple-access and broadcast with a common message) half-duplex two-way channel is derived in [5], [6] for Gaussian and discrete memoryless channels, respectively. The second channel model we consider is the two-way linear deterministic Z channel. The capacity region of the one-way deterministic Z channel is found in [7].

The last channel model considered is the two-way linear deterministic IC. The capacity region of the one-way modulo 2 adder IC is known [8] and is a special example of a more general class of deterministic IC for which capacity is known [9], including the one-way linear deterministic IC [4]. The work here is also related to one-way ICs with perfect output feedback [10], [11], with rate-limited feedback [12] and with noisy, interfering feedback [13]. In all these channel models only two messages are present and the backwards feedback links, whether perfect, noisy, or interfering still serve only to further rates in the forward direction. The tradeoff between sending new information versus feedback on each of the links is not addressed; the only other example of such a 4-message two-way interference channel besides our prior work [14], [15] is in Section VI of [13], where an example of a linear deterministic scheme in a specific regime is provided.

The two-way versions of the binary modulo 2 MAC/BC, Z and IC were first considered in [14], where adaptation was shown to be useless – time-sharing achieved capacity. Here we extend the limited results on multi-user two-way channels to the more general class of linear deterministic channel models.

II. Models, Definitions and Notations

We introduce three multi-user two-way linear deterministic channels (we drop the "linear deterministic" from now on) which are shown in Fig. 1, where we see that all nodes act as both transmitters (encoders) and receivers (decoders), and let M_{jk} denote the message from node j to node k:

• *the two-way MAC/BC channel:* transmitter 1 and 3 send independent messages M_{12} and M_{32} to receiver 2, respectively, forming a Multiple Access Channel (MAC) in the $\rightarrow$ direction. Meanwhile, transmitter 2 sends two independent messages M_{21} and M_{23} to receiver 1 and 3, respectively, forming a Broadcast Channel (BC) in the $\leftarrow$ direction.

• *the two-way Z channel:* transmitter 1 and 4 send messages M_{12} and M_{43} to receiver 2 and 3 respectively. Transmitter 2 and 3 send messages (M_{21}, M_{23}) and (M_{32}, M_{34}) to receivers 1,3 and 2,4 respectively. We thus have two Z channels in opposite directions.

• *the two-way interference channel:* transmitter 1 and 3 send messages M_{12} and M_{34} to receiver 2 and 4, respectively, forming an IC in the $\rightarrow$ direction. Similarly, transmitter 2 and 4 send messages M_{21} and M_{43} to receiver 1 and 3 respectively, forming another IC in the $\leftarrow$ direction.

The channel inputs and outputs of user $j \in \{1,2,3,4\}$ at discrete time i are $X_{j,i}$ and $Y_{j,i}$. Let $A_j^i = (A_{j,1}, A_{j,2}, ..., A_{j,i})$, for any given time i. A node is said to employ *adaptation* (rather than "feedback" to emphasize *two-way* communications rather than one-way communication with "feedback" links. One may equivalently use the term "cooperation", though we feel "adaptation" emphasizes the fact that a node can adapt current inputs to past outputs.) if the channel input at time i is a function of the previously received outputs, $X_{j,i} = f_j(M_{jk}, Y_j^{i-1})$, where f_j ($j \in \{1,2,3,4\}$) are deterministic functions. If a node behaves in a *non-adaptive* or *restricted* fashion then its inputs are functions of its messages only, i.e. $X_{j,i} = f_j(M_{jk})$. If some nodes adapt while the others do not, we refer this as *partial adaptation*, and will specify which nodes adapt.

The messages M_{jk} of rate R_{jk} are uniformly distributed in $\{1, 2, \cdots 2^{nR_{jk}}\}$ for j,k in the appropriate sets depending on the channel model. At each time step $0 \le i \le n$, for n the blocklength, encoder j selects the next input $X_{j,i}(M_{jk}, Y_j^{i-1})$ (which may also be a function of 2 messages in the Z channel). The channel inputs and outputs are binary vectors, and all addition will be bit-wise and modulo 2. We furthermore let S denote an $N \times N$ lower shift matrix, where N will be defined for each channel model. In addition, we define $n_{jk} = \lfloor \log h_{jk}^2 P_j \rfloor$ to indicate the number of signal bit levels from transmitter j to receiver k, where h_{jk} is the channel gains and P_j denotes the power of the transmitter j. Note that n_{ii} (the interference caused at a node due to its own transmission) exists in our model, though we may just as well have left it out given the additive nature of the model; including/leaving it out is not WLOG for other channel models such as the binary multiplier channel. Note that Receiver k uses a decoding function $g_k : \mathcal{Y}_k^n \rightarrow \widehat{\mathcal{M}}_{jk}$ to obtain an estimate $\widehat{M}_{jk}$ of the transmitted message M_{jk}. The capacity region is the supremum over all rate tuples which simultaneously drive the probability that any of the estimated messages is not equal to the true message, to zero as $n \rightarrow \infty$. We now proceed to define the channel model for each different channel and obtain its capacity region.

III. The Capacity Region of the Two-way Linear Deterministic MAC/BC

The two-way linear deterministic MAC/BC channel is defined by the following input/output equations as in Fig. 1(a). All nodes are permitted to adapt, so that at channel use i, $X_{1,i} = f_1(M_{12}, Y_1^{i-1})$, $X_{2,i} = f_2(M_{21}, M_{23}, Y_2^{i-1})$, and $X_{3,i} = f_3(M_{32}, Y_3^{i-1})$. In this case, the capacity region maybe stated as follows:

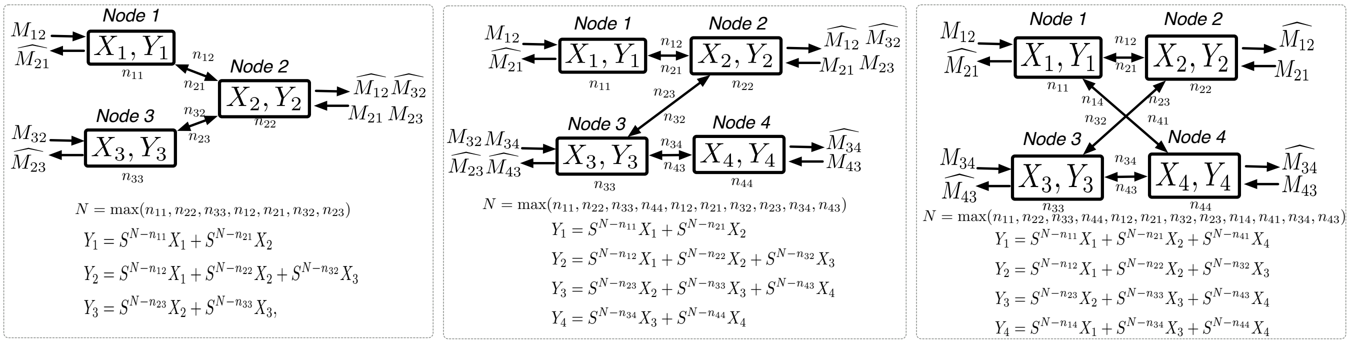
Theorem 1: The capacity region of the two-way linear deterministic MAC/BC is the set of non-negative rate tuples $(R_{12}, R_{32}, R_{21}, R_{23})$ such that

$$\text{MAC } \rightarrow \begin{cases} R_{12} \le n_{12}, \quad R_{32} \le n_{32}, \\ R_{12} + R_{32} \le \max(n_{12}, n_{32}) \end{cases} \tag{1}$$

$$\text{BC } \leftarrow \begin{cases} R_{21} \le n_{21}, \quad R_{23} \le n_{23} \\ R_{21} + R_{23} \le \max(n_{21}, n_{23}). \end{cases} \tag{2}$$

Proof: Achievability may be argued via [3] by mimicking a one-way MAC and one-way BC channel in opposite directions and noting that in this channel model, each user may subtract off its own transmitted signal from its received signal. We note that for this channel, the bounds may be obtained by the cut-set outer bound, but that we derive it in an alternative way nonetheless as the technique illustrates how adaptation may be taken into account and leads to more general outer bounds (for two-way MAC/BC channels), which are omitted.

$$\begin{aligned} & n(R_{12} - \epsilon) \le I(M_{12}; Y_2^n | M_{21}, M_{23}, M_{32}) \\ & \le \sum_{i=1}^{n} [H(Y_{2,i} | Y_2^{i-1}, M_{21}, M_{23}, M_{32}, X_2^i)] \\ & \overset{(a)}{\le} \sum_{i=1}^{n} [H(Y_{2,i} | Y_2^{i-1}, M_{21}, M_{23}, M_{32}, X_2^i, X_3^i)] \\ & \le \sum_{i=1}^{n} [H(S^{N-n_{12}} X_{1,i})] \le n(n_{12}), \end{aligned}$$



(a) Two-way linear deterministic MAC/BC (b) Two-way linear deterministic Z channel (c) Two-way linear deterministic interference channel

Fig. 1. Three two-way linear deterministic channel models under consideration.

where (a) follows from a Markov chain that given M_{32}, X_2^i, we have X_3^i. The other single rate bounds follow similarly.

$$
\begin{aligned}
&n(R_{12}+R_{32}-\epsilon) \leq I(M_{12}, M_{32}; Y_2^n | M_{21}, M_{23}) \\
&\leq \sum_{i=1}^{n} [H(Y_{2,i}|Y_2^{i-1}, M_{21}, M_{23}, X_2^i)] \\
&\leq \sum_{i=1}^{n} [H(S^{N-n_{12}}X_{1,i} + S^{N-n_{32}}X_{3,i})] \leq n(\max(n_{12}, n_{32})).
\end{aligned}
$$

We can obtain the other sum-rate bound in a similar way. ■

Remark 2: Without adaptation, the channel would correspond to a MAC channel simultaneously transmitting with a BC channel with restricted nodes. Since we are able to achieve the desired rates in one channel use, adaptation is useless, and the capacity region is a four dimensional region that is equivalent to the capacity region of the linear deterministic MAC and the linear deterministic BC in opposite directions.

IV. The Capacity Region of the Two-way Linear Deterministic Z Channel

The linear deterministic two-way Z channel is defined by the input / output equations in Fig. 1(b). If all nodes employ adaptation, then at channel use i, $X_{1,i} = f_1(M_{12}, Y_1^{i-1})$, $X_{2,i} = f_2(M_{21}, M_{23}, Y_2^{i-1})$, $X_{3,i} = f_3(M_{32}, M_{34}, Y_3^{i-1})$, $X_{4,i} = f_4(M_{43}, Y_4^{i-1})$.

Theorem 3: The capacity region of the two-way linear deterministic Z channel is the set of all rate-tuples $(R_{12}, R_{21}, R_{23}, R_{32}, R_{34}, R_{43})$ which satisfy the following:

$$
\text{Z} \rightarrow \begin{cases} R_{12} \leq n_{12}, \quad R_{32} \leq n_{32}, \quad R_{34} \leq n_{34} \\ R_{12}+R_{32} \leq \max(n_{12}, n_{32}) \\ R_{32}+R_{34} \leq \max(n_{32}, n_{34}) \\ R_{12}+R_{32}+R_{34} \leq \max(n_{12}, n_{32}) + [n_{34}-n_{32}]^+ \end{cases}
$$

$$
\text{Z} \leftarrow \begin{cases} R_{43} \leq n_{43}, \quad R_{23} \leq n_{23}, \quad R_{21} \leq n_{21} \\ R_{43}+R_{23} \leq \max(n_{43}, n_{23}) \\ R_{23}+R_{21} \leq \max(n_{23}, n_{21}) \\ R_{43}+R_{23}+R_{21} \leq \max(n_{43}, n_{23}) + [n_{21}-n_{23}]^+. \end{cases}
$$

Proof: We first note that the capacity of a class of deterministic Z channels is shown in [7, Th. 3.1]. To show achievability of the above, we use the achievability scheme of [7, Th. 3.1] in each $\rightarrow$ and $\leftarrow$ direction with non-adaptive nodes (adaptive may mimic non-adaptive). By making the appropriate correspondences, we see that the above is achievable and equivalent to two one-way Z channels.

Now, we prove the converse. We note that again, all but the triple-rate bounds may be obtained by the two-way cut-set bound, but that they are left as they illustrate the impact of adaptation and how these bounds made be generalized. By symmetry, we only show two sum-rate bounds.

$$
\begin{aligned}
&n(R_{32}+R_{34}-\epsilon) \leq I(M_{32}, M_{34}; Y_2^n, Y_4^n | M_{21}, M_{23}, M_{43}, M_{12}) \\
&\leq \sum_{i=1}^{n} [H(Y_{2,i}, Y_{4,i}| M_{21}, M_{23}, M_{43}, M_{12}, Y_2^{i-1}, Y_4^{i-1}, X_2^i, X_4^i)] \\
&\overset{(a)}{\leq} \sum_{i=1}^{n} [H(Y_{2,i}, Y_{4,i}| M_{21}, M_{23}, M_{43}, M_{12}, Y_2^{i-1}, Y_4^{i-1}, X_2^i, X_4^i, X_1^i)] \\
&\leq \sum_{i=1}^{n} [H(S^{N-n_{32}}X_{3,i}, S^{N-n_{34}}X_{3,i})] \leq n(\max(n_{32}, n_{34})),
\end{aligned}
$$

where (a) follows from a Markov chain that given M_{12}, X_2^i, we have X_1^i. We also have,

$$
\begin{aligned}
&n(R_{12}+R_{32}+R_{34}-\epsilon) \leq I(M_{12}; Y_2^n | M_{21}, M_{23}, M_{43}) \\
&+ I(M_{32}, M_{34}; Y_2^n | M_{43}, M_{12}, M_{21}, M_{23}) \\
&\quad + I(M_{32}, M_{34}; Y_4^n | M_{43}, M_{12}, M_{21}, M_{23}, Y_2^n) \\
&\leq H(Y_2^n | M_{21}, M_{23}, M_{43}) - H(Y_2^n | M_{12}, M_{21}, M_{23}, M_{43}) \\
&+ H(Y_2^n | M_{12}, M_{21}, M_{23}, M_{43}) + H(Y_4^n | M_{43}, M_{12}, M_{21}, M_{23}, Y_2^n) \\
&\leq \sum_{i=1}^{n} [H(Y_{2,i}|Y_2^{i-1}, M_{21}, M_{23}, X_2^i) \\
&\quad + H(Y_{4,i}| M_{12}, M_{21}, M_{23}, M_{43}, Y_4^{i-1}, Y_2^n)] \\
&\overset{(b)}{\leq} \sum_{i=1}^{n} [H(S^{N-n_{12}}X_{1,i} + S^{N-n_{32}}X_{3,i}) \\
&\quad + H(S^{N-n_{34}}X_{3,i} | M_{12}, M_{21}, M_{23}, M_{43}, Y_4^{i-1}, \\
&\quad X_4^i, S^{N-n_{12}}X_{1,i} + S^{N-n_{22}}X_{2,i} + S^{N-n_{32}}X_{3,i}, X_2^n, X_1^n)] \\
&\leq \sum_{i=1}^{n} [H(S^{N-n_{12}}X_{1,i} + S^{N-n_{32}}X_{3,i}) + H(S^{N-n_{34}}X_{3,i} | S^{N-n_{32}}X_{3,i})] \\
&\leq n(\max(n_{12}, n_{32}) + [n_{34}-n_{32}]^+).
\end{aligned}
$$

In (b), given M_{12} and X_2^n, we may construct X_1^n. ■

Remark 4: Again, we are always able to achieve the desired rates in Theorem 3 in only one channel use, therefore adaptation is useless. The capacity region of this channel, a six dimensional region, is exactly equivalent to the capacity region of the two one-way linear deterministic Z channels.

V. The Capacity Region of the Two-way Linear Deterministic Interference Channel

The two-way linear deterministic IC is defined by the input / output equations in Fig. 1(c). Now we define partial adaptation (fixed nodes 1 and 3: "restricted nodes") as:

$$X_{1,i} = f_1(M_{12}), \quad X_{2,i} = f_2(M_{21}, Y_2^{i-1}) \tag{3}$$

$$X_{3,i} = f_3(M_{34}), \quad X_{4,i} = f_4(M_{43}, Y_4^{i-1}) \tag{4}$$

We first prove a Lemma regarding partial adaptation, which is key in showing that partial adaptation is useless, and that the inability of *certain* nodes to adapt essentially "blocks" the ability of adaptation to help at all.

Lemma 5: Under the partial adaptation conditions in (3) – (4), for some deterministic functions f_5 and f_6,

$$X_{2,i} = f_5(M_{12}, M_{21}, M_{34}) \perp M_{43}, \quad \forall i \tag{5}$$

$$X_{4,i} = f_6(M_{43}, M_{34}, M_{12}) \perp M_{21}, \quad \forall i \tag{6}$$

where $\perp$ denotes independence.

Proof: Note that $X_{2,i} = f_2(M_{21}, Y_2^{i-1})$ and $Y_2^{i-1} = S^{N-n_{12}}X_1^{i-1} + S^{N-n_{22}}X_2^{i-1} + S^{N-n_{32}}X_3^{i-1}$. Since X_1^{i-1} and X_3^{i-1} are functions only of M_{12} and M_{34} respectively, we may conclude that there exists a function f^* such that $X_{2,i} = f^*(M_{21}, M_{12}, M_{34}, X_2^{i-1})$. Iterating this argument, and noting that $X_{2,1}$ is only a function of M_{21}, we obtain the theorem. The result for $X_{4,i}$ follows by a similar argument. ■

Theorem 6: The capacity region of the two-way linear deterministic IC is the set of $(R_{12}, R_{21}, R_{34}, R_{43})$ which satisfy the equations in (A) and (B) in Fig. 2.

Proof: For achievability, we consider the two-way interference channel as two one-way interference channels and apply the well-known Han-Kobayashi scheme [16] to achieve the inner bound (ignore the ability to adapt). Now we prove the converse. Due to space constraints we only focus on the sum-rates; single-rates follow as in (1), and using Lemma 5.

$$\begin{aligned}
&n(R_{12}+R_{34}-\epsilon)\\
&\overset{(a)}{\leq} I(M_{12};Y_2^n|M_{21},M_{43}) + I(M_{34};Y_4^n,Y_2^n|M_{12},M_{21},M_{43})\\
&\leq I(M_{12};Y_2^n|M_{21},M_{43}) + I(M_{34};Y_2^n|M_{21},M_{12},M_{43})\\
&+ H(Y_4^n|M_{21},M_{12},M_{43},Y_2^n)\\
&\overset{(b)}{=} I(M_{12};Y_2^n|M_{21},M_{43}) + I(M_{34};Y_2^n|M_{21},M_{12},M_{43})\\
&+ \sum_{i=1}^n [H(S^{N-n_{34}}X_{3,i}|M_{21},M_{12},M_{43},Y_4^{i-1},X_4^i,Y_2^n,X_2^n,X_1^i)]\\
&\leq \sum_{i-1}^n [H(Y_{2,i}|Y_2^{i-1},M_{21},X_2^i) - H(Y_{2,i}|Y_2^{i-1},M_{12},M_{21},M_{43})\\
&\quad + H(Y_{2,i}|Y_2^{i-1},M_{12},M_{21},M_{43})\\
&+ H(S^{N-n_{34}}X_{3,i}|M_{21},M_{12},M_{43},Y_4^{i-1},X_4^i,\\
&\quad S^{N-n_{12}}X_1^n + S^{N-n_{22}}X_2^n + S^{N-n_{32}}X_3^n, X_2^n, X_1^i)]\\
&\leq \sum_{i=1}^n [H(S^{N-n_{12}}X_{1,i} + S^{N-n_{32}}X_{3,i})\\
&\quad + H(S^{N-n_{34}}X_{3,i}|S^{N-n_{32}}X_{3,i})]\\
&\leq n(\max(n_{12},n_{32}) + [n_{34}-n_{32}]^+) \overset{(c)}{=} n(\max(p,q) + [p-q]^+)
\end{aligned}$$

We introduce the genie Y_2^n in the second mutual information term in (a), i.e. we provide asymmetric side information to only one receiver. In (b), we add X_1^i in the entropy term because of the iterated argument that, given M_{12}, X_2^n, X_4^i, we can construct X_1^i. For (c), we consider the symmetric case with $p = n_{12} = n_{21} = n_{34} = n_{43}, q = n_{14} = n_{41} = n_{23} = n_{32}$.

Remark 7: We do **not** need partial adaptation in this bound, and so these conclusions actually hold for **full adaptation.** It may further be shown that, assuming symmetry, adaptation is useless when two-way interference is strong ($\alpha \geq 1, \alpha = q/p$) and weak in some interval ($2/3 \leq \alpha \leq 1, \alpha = q/p$). Interestingly, when $2/3 \leq \alpha \leq 2$, the "V" curve is also the capacity for the linear deterministic symmetric IC with feedback [11]. Adding an asymmetric genie Y_4^n in the first term in (a), yields the second sum-rate bound in Fig. 2 (A).

$$\begin{aligned}
&n(R_{12}+R_{34}-\epsilon) \leq I(M_{12};Y_2^n,S^{N-n_{14}}X_1^n,M_{21},M_{43})\\
&\quad + I(M_{34};Y_4^n,S^{N-n_{32}}X_3^n,M_{21},M_{43})\\
&\overset{(d)}{=} H(Y_2^n|S^{N-n_{14}}X_1^n,M_{43},M_{21}) + H(S^{N-n_{14}}X_1^n|M_{43},M_{21})\\
&- H(Y_2^n,S^{N-n_{14}}X_1^n|M_{12},M_{21},M_{43})\\
&+ H(Y_4^n|S^{N-n_{32}}X_3^n,M_{43},M_{21}) + H(S^{N-n_{32}}X_3^n|M_{43},M_{21})\\
&- H(Y_4^n,S^{N-n_{32}}X_3^n|M_{34},M_{21},M_{43})\\
&\overset{(e)}{=} H(Y_2^n|S^{N-n_{14}}X_1^n,M_{43},M_{21})\\
&+ H(S^{N-n_{14}}X_1^n|M_{43},M_{21},M_{34}) - H(S^{N-n_{32}}X_3^n|M_{12},M_{43},M_{21})\\
&+ H(Y_4^n|S^{N-n_{32}}X_3^n,M_{43},M_{21})\\
&+ H(S^{N-n_{32}}X_3^n|M_{43},M_{21},M_{12}) - H(S^{N-n_{14}}X_1^n|M_{43},M_{21},M_{34})\\
&= H(Y_2^n|S^{N-n_{14}}X_1^n,M_{43},M_{21}) + H(Y_4^n|S^{N-n_{32}}X_3^n,M_{43},M_{21})\\
&\leq \sum_{i=1}^n [H(S^{N-n_{12}}X_{1,i} + S^{N-n_{32}}X_{3,i}|S^{N-n_{14}}X_{1,i})\\
&+ H(S^{N-n_{14}}X_{1,i} + S^{N-n_{34}}X_{4,i}|S^{N-n_{32}}X_{3,i})]\\
&\leq n(\max([n_{12}-n_{14}]^+,n_{32}) + \max([n_{34}-n_{32}]^+,n_{14})),
\end{aligned}$$

where (d) follows from the independence of the messages. For (e), the 2nd and 5th terms follow since X_1 and X_3 are functions only of M_{12} and M_{34}. The 3rd and 6th terms follow for the same reason, together with Lemma 5.

$$\begin{aligned}
&n(R_{21}+R_{43}-\epsilon) \leq I(M_{21};Y_1^n,S^{N-n_{23}}X_2^n,M_{12},M_{34})\\
&+ I(M_{43};Y_3^n,S^{N-n_{41}}X_4^n,M_{12},M_{34})\\
&\overset{(f)}{=} H(Y_1^n|S^{N-n_{23}}X_2^n,M_{12},M_{34}) + H(S^{N-n_{23}}X_2^n|M_{12},M_{34})\\
&- H(Y_1^n,S^{N-n_{23}}X_2^n|M_{12},M_{34},M_{21})\\
&+ H(Y_3^n|S^{N-n_{41}}X_4^n,M_{12},M_{34}) + H(S^{N-n_{41}}X_4^n|M_{12},M_{34})\\
&- H(Y_3^n,S^{N-n_{41}}X_4^n|M_{12},M_{34},M_{43})\\
&\overset{(g)}{=} H(Y_1^n|S^{N-n_{23}}X_2^n,M_{12},M_{34}) + H(S^{N-n_{23}}X_2^n|M_{12},M_{34},M_{43})\\
&- H(S^{N-n_{41}}X_4^n|M_{12},M_{34},M_{21})
\end{aligned}$$

$R_{12} \le n_{12}, \quad R_{34} \le n_{34},$
$R_{12} + R_{34} \le \max(n_{12}, n_{32}) + [n_{34} - n_{32}]^+$
$R_{12} + R_{34} \le \max(n_{34}, n_{14}) + [n_{12} - n_{14}]^+$
$R_{12} + R_{34} \le \max([n_{12} - n_{14}]^+, n_{32}) + \max([n_{34} - n_{32}]^+, n_{14})$
$2R_{12} + R_{34} \le \max(n_{12}, n_{32}) + [n_{12} - n_{14}]^+ + \max([n_{34} - n_{32}]^+, n_{14})$
$R_{12} + 2R_{34} \le \max(n_{34}, n_{14}) + [n_{34} - n_{32}]^+ + \max([n_{12} - n_{14}]^+, n_{32})$

(A) IC in $\rightarrow$ direction

$R_{21} \le n_{21}, \quad R_{43} \le n_{43}$
$R_{21} + R_{43} \le \max(n_{21}, n_{41}) + [n_{43} - n_{41}]^+$
$R_{21} + R_{43} \le \max(n_{43}, n_{23}) + [n_{21} - n_{23}]^+$
$R_{21} + R_{43} \le \max([n_{21} - n_{23}]^+, n_{41}) + \max([n_{43} - n_{41}]^+, n_{23})$
$2R_{21} + R_{43} \le \max(n_{21}, n_{41}) + [n_{21} - n_{23}]^+ + \max([n_{43} - n_{41}]^+, n_{23})$
$R_{21} + 2R_{43} \le \max(n_{43}, n_{23}) + [n_{43} - n_{41}]^+ + \max([n_{21} - n_{23}]^+, n_{41})$

(B) IC in $\leftarrow$ direction

Fig. 2. Capacity region of two-way linear deterministic interference channel with partial adaptation.

$$
\begin{aligned}
&+ H(Y_3^n | S^{N-n_{41}} X_4^n, M_{12}, M_{34}) + H(S^{N-n_{41}} X_4^n | M_{12}, M_{34}, M_{21}) \\
&- H(S^{N-n_{23}} X_2^n | M_{12}, M_{34}, M_{43}) \\
&\le \sum_{i=1}^n [H(S^{N-n_{21}} X_{2,i} + S^{N-n_{41}} X_{4,i} | S^{N-n_{23}} X_{2,i}) \\
&+ H(S^{N-n_{43}} X_{4,i} + S^{N-n_{23}} X_{2,i} | S^{N-n_{41}} X_{4,i})] \\
&\le n(\max([n_{21} - n_{23}]^+, n_{41}) + \max([n_{43} - n_{41}]^+, n_{23})),
\end{aligned}
$$

where (f) follows from the independence of the messages. Equation (g) follows from partial adaptation and Lemma 5.

Remark 8: In the above, nodes 1 and 3 were restricted (needed for Lemma 5). By symmetry, we may obtain the same result if nodes 2 and 4 were restricted. Finally,

$$
\begin{aligned}
&n(2R_{12} + R_{34} - \epsilon) \\
&\le I(M_{12}; Y_2^n | M_{21}, M_{43}) + I(M_{12}; Y_2^n, Y_4^n | M_{21}, M_{43}, M_{34}) \\
&+ I(M_{34}; Y_4^n, S^{N-n_{32}} X_3^n, M_{21}, M_{43}) \\
&\overset{(h)}{=} H(Y_2^n | M_{21}, M_{43}) - H(Y_2^n | M_{21}, M_{43}, M_{12}) \\
&+ H(Y_4^n | M_{21}, M_{43}, M_{34}) + H(Y_2^n | M_{21}, M_{43}, M_{34}, Y_4^n) \\
&+ H(Y_4^n, S^{N-n_{32}} X_3^n | M_{21}, M_{43}) \\
&- H(Y_4^n, S^{N-n_{32}} X_3^n | M_{34}, M_{21}, M_{43}) \\
&\overset{(i)}{=} H(Y_2^n | M_{21}, M_{43}) - H(Y_2^n | M_{21}, M_{43}, M_{12}) \\
&+ H(S^{N-n_{32}} X_3^n | M_{43}, M_{21}, M_{12}) \\
&+ H(Y_4^n | S^{N-n_{32}} X_3^n, M_{43}, M_{21}) + H(Y_4^n | M_{21}, M_{43}, M_{34}) \\
&- H(Y_4^n, S^{N-n_{32}} X_3^n | M_{34}, M_{21}, M_{43}) + H(Y_2^n | M_{21}, M_{43}, M_{34}, Y_4^n) \\
&\le \sum_{i=1}^n [H(S^{N-n_{12}} X_{1,i} + S^{N-n_{32}} X_{3,i}) + H(S^{N-n_{14}} X_{1,i} \\
&+ S^{N-n_{34}} X_{3,i} | S^{N-n_{32}} X_{3,i}) + H(S^{N-n_{12}} X_{1,i} | S^{N-n_{14}} X_{1,i})] \\
&= n(\max(n_{12}, n_{32}) + \max([n_{34} - n_{32}]^+, n_{14}) + [n_{12} - n_{14}]^+),
\end{aligned}
$$

where (h) follows from the independence of the messages, and (i) from the definition of partial adaptation and Lemma 5. We may similarly prove the other bounds of this form. ■

Remark 9: We again see that, under partial adaptation constraints, adaptation is useless and we obtain the capacity of two one-way ICs. Essentially, this partial adaptation prevented messages being relayed by other messages (which was also impossible in the MAC/BC and Z channels). For example, under full adaptation, message M_{12} may be relayed from Tx1 to Rx 2 through nodes 3 and 4. This path is "blocked" by the partial adaptation assumption, as node 3 could not adapt to carry M_{12}. However, it should be pointed out that this is not necessary in general: adaptation in the two-way modulo 2 adder IC is useless [14], but the path is not blocked.

VI. Conclusion

We obtained the capacity regions of the two-way linear deterministic MAC/BC channel, the two-way linear deterministic Z channel, both with full adaptation, and the two-way linear deterministic IC, with partial adaptation. Interestingly, adaptation is not needed to attain the capacity regions even though it is permitted, demonstrating *multi-user* examples of two-way channels where adaptation or interaction is useless.

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