On Gaussian Interference Channels with Mixed Gaussian and Discrete Inputs

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***Abstract*—This paper studies the sum-rate of a class of memoryless, real-valued additive white Gaussian noise interference channels (IC) achievable by treating interference as noise (TIN). We develop and analytically characterize the rates achievable by a new strategy that uses superpositions of Gaussian and discrete random variables as channel inputs. Surprisingly, we demonstrate that TIN is sum-generalized degrees of freedom optimal and can achieve to within an additive gap of $O(1)$ or $O(\log\log(\mathbf{SNR}))$ to the symmetric sum-capacity of the classical IC. We also demonstrate connections to other channels such as the IC with partial codebook knowledge and the block asynchronous IC.**

I. Introduction

Consider a memoryless real-valued additive white Gaussian noise interference channel with input-output relationship

$$Y_1^n = h_{11}X_1^n + h_{12}X_2^n + Z_1^n, \tag{1a}$$

$$Y_2^n = h_{21}X_1^n + h_{22}X_2^n + Z_2^n, \tag{1b}$$

where $X_j^n := (X_{j1}, \cdots X_{jn})$ and $Y_j^n := (Y_{j1}, \cdots Y_{jn})$ are the length-n vector inputs and outputs, respectively, for user $j \in [1:2]$, the noise vectors Z_j^n are independent and have i.i.d. zero-mean unit-variance Gaussian components and the inputs X_j^n is subject to a per-block power constraint $\frac{1}{n}\sum_{i=1}^n X_{ji}^2 \le 1$, for $j \in [1:2]$. Input X_j^n, $j \in [1:2]$, is a function of the independent message W_j that is uniformly distributed on $[1:2^{nR_j}]$, where R_j is the rate and n the block length. Receiver $j \in [1:2]$ wishes to recover W_j from the channel output Y_j^n with arbitrary small probability of error. Achievable rates and capacity region are defined in the usual way [1].

In this work we shall focus on the sum-capacity, defined as the largest achievable $R_1 + R_2$. It was shown by [2] that the sum-capacity of any information stable [3] IC is given by

$$\max_{n\in\mathbb{N}, P_{X_1^n,X_2^n}=P_{X_1^n}P_{X_2^n}} \frac{1}{n}\Big(I(X_1^n;Y_1^n) + I(X_2^n;Y_2^n)\Big). \tag{2}$$

For the Gaussian noise channel in (1), the maximization in (2) is further restricted to inputs satisfying the power constraint.

For sake of simplicity, we shall focus from now on on the *symmetric* Gaussian IC only, defined as $|h_{11}|^2 = |h_{22}|^2 = \mathsf{S} \ge 0$, $|h_{12}|^2 = |h_{21}|^2 = \mathsf{I} \ge 0$, and denote its sum-capacity in (2) as $C(\mathsf{S},\mathsf{I})$. In general, little is known about the optimizing distribution in (2) and only some special cases have been solved. In [6] it was showed that i.i.d. Gaussian inputs maximize (2) for $\sqrt{\frac{\mathsf{I}}{\mathsf{S}}}(1+\mathsf{I}) \le \frac{1}{2}$. In contrast, the authors of [7] show that in general Gaussian inputs do not maximize expressions of the form of (2). The difficulty of the problem in (2) arises from the competitive nature of the problem [8]: for example, say X_2 is i.i.d. Gaussian, taking X_1 to be Gaussian increases $I(X_1^n;Y_1^n)$ but simultaneously decreases $I(X_2^n;Y_2^n)$, as Gaussians are known to be the "best" inputs for Gaussian point-to-point channels, but are also the "worst" type of noise for a Gaussian input.

A lower bound to the sum-capacity can be obtained by considering i.i.d inputs in (2), thus giving

$$R_L(\mathsf{S},\mathsf{I}) = \max_{P_{X_1,X_2}=P_{X_1}P_{X_2}} \Big(I(X_1;Y_1) + I(X_2;Y_2)\Big), \tag{3}$$

where the maximization in (3) is further restricted to inputs satisfying the power constraint. In [8], [5] the authors demonstrated the existence of input distributions that outperform i.i.d. Gaussian inputs in $R_L(\mathsf{S},\mathsf{I})$ for certain asynchronous IC. Both works use local perturbations of an i.i.d. Gaussian input: [5, Lemma 3] considers a fourth order approximation of mutual information, while [8, Theorem 4] uses perturbation in the direction of Hermite polynomials of order larger than three. In both cases the input distribution is assumed to have a density. For the cases reported in [5], [8], the improvement over i.i.d. Gaussian inputs shows in the decimal digits of the achievable rates; it is hence not clear that these inputs can actually provide substantial rate gains compared to Gaussian inputs.

Recently in [9], for the IC with one oblivious receiver, we showed that a properly chosen discrete input has a somewhat different behavior than continuous inputs: it may yield a "good" $I(X_1;Y_1)$ while keeping $I(X_2;Y_2)$ relatively unchanged, thus substantially improving the rates compared to using Gaussian inputs in the same achievable region expression. In this work we seek to analytically evaluate the lower bound in (3) for a special class of *mixed Gaussian and discrete inputs* by generalizing the approach of [9].

Our contributions, and paper organization, are as follows. In Section II we present the main tools used in our analysis: a lower bound on the the mutual information attained by a discrete input on a point-to-point Gaussian noise channel and tools to compute the cardinality and minimum distances of sum-sets. In Section III we present an achievable sum-rate valid for any memoryless IC obtained by evaluating $R_L(\mathsf{S},\mathsf{I})$ with an input that consists of the superposition of discrete and Gaussian components, which we term *mixed input* from now on. In Section IV we show that, in terms of generalized degrees of freedom (gDoF), mixed inputs in $R_L(\mathsf{S},\mathsf{I})$ achieve the optimal $C(\mathsf{S},\mathsf{I})$. In Section V we show that at any finite (S,I), $R_L(\mathsf{S},\mathsf{I})$ with mixed inputs lies to within a constant gap of $O(1)$ or of $\log(\log(\mathsf{S}))$ from $C(\mathsf{S},\mathsf{I})$. In Section VI we apply our results to the block asynchronous IC and the oblivious IC. Section VII concludes the paper.

We use the following notation convention: if A is a random variable (r.v.) we denote its support by $\mathsf{supp}(A)$; $|A|$ is the cardinality of a set A or cardinality of $\mathsf{supp}(A)$ if A is r.v.; the symbol $d_{\min(\mathcal{S})} = \min_{i\neq j: s_i,s_j\in\mathcal{S}} |s_i - s_j|$ denotes the minimum distance among the points in the set $\mathcal{S}$. With some abuse of notation we also use $d_{\min(A)}$ to denote $d_{\min(\mathsf{supp}(A))}$ for a r.v. A; $X \sim \mathcal{N}(\mu,\sigma^2)$ denotes a real-valued Gaussian r.v. X with mean μ and variance σ^2; $\log(\cdot)$ denotes logarithms in base 2 and $\ln(\cdot)$ in base e; we let $[x]^+ := \max(x,0)$ and $\log^+(x) := [\log(x)]^+$; $\lfloor x\rfloor$ refers to largest integer less than or equal to x; $\mathrm{PAM}(N,d_{\min})$ denotes the uniform distribution over a zero-mean real-valued Pulse Amplitude Modulation (PAM) constellation with N points and minimum distance $d_{\min}$ (and average energy $\mathcal{E} = d_{\min}^2 \frac{N^2-1}{12}$).

In the following we will compare $R_L(\mathsf{S},\mathsf{I})$ to an upper bound on $C(\mathsf{S},\mathsf{I})$, denoted by $R_U(\mathsf{S},\mathsf{I})$, from the classical IC with full codebook knowledge at all nodes and with block synchronous communication. $R_U(\mathsf{S},\mathsf{I})$ is therefore an upper bound to the capacity of block asynchronous or IC with partial codebook knowledge. As mentioned earlier, from [6] we have

$$R_L(\mathsf{S},\mathsf{I}) = C(\mathsf{S},\mathsf{I}) = \log\left(1+\frac{\mathsf{S}}{1+\mathsf{I}}\right) \text{ if } \sqrt{\frac{\mathsf{I}}{\mathsf{S}}}(1+\mathsf{I}) \le \frac{1}{2}.$$

Moreover, $C(\mathsf{S},\mathsf{I}) \le R_U(\mathsf{S},\mathsf{I}) := \min(O_1,O_2,O_3)$ with

$$O_1 := \log(1+\mathsf{S}) \text{ cut-set bound,} \tag{4a}$$

$$O_2 := \log\left(1+\mathsf{I}+\frac{\mathsf{S}}{1+\mathsf{I}}\right) \text{ from [10],} \tag{4b}$$

$$2O_3 := \log^+\left(\frac{1+\mathsf{S}}{1+\mathsf{I}}\right) + \log(1+\mathsf{I}+\mathsf{S}) \text{ from [11].} \tag{4c}$$

II. Main Tools

In this Section we present the main tools to evaluate the lower bound in (3) under mixed inputs. At the core of our proof is the following new lower bound on the rate achieved by a discrete input on a point-to-point Gaussian noise channel.

Theorem 1 ([9, Theorem 1])**.** *Let X_D be a discrete r.v. with N distinct masses and minimum distance $d_{\min}$, $Z_G \sim \mathcal{N}(0,1)$, and S be a non-negative constant. Then,*

$$I(X_D; \sqrt{\mathsf{S}}X_D + Z_G) \ge \mathsf{I_d}\left(N, \frac{\mathsf{S}d_{\min}^2}{4}\right), \tag{5}$$

where, for $N\in\mathbb{N}$ and $x\in\mathbb{R}$ we define

$$\mathsf{I_d}(N,x) := \left[\log(N) - \frac{1}{2}\log\left(\frac{\mathrm{e}}{2}\right) - \log\left(1+(N-1)\mathrm{e}^{-x}\right)\right]^+.$$

The inequality in (5) nicely captures the effect of the cardinality and minimum distance of the discrete constellation on the achievable mutual information. We are not the first to analytically consider discrete inputs. For point-to-point Gaussian noise channel: [12] characterized the optimal discrete input distribution at high and low SNR; [13] found tight high-SNR asymptotic of mutual information for any discrete constellation whose size N is independent of SNR; in [14] the authors considered the point-to-point power-constrained Gaussian noise channel and derived lower bounds on the achievable rate when the input is constrained to be PAM. However, in our work we need a *firm* lower bound that holds for *all* SNR's and all input distributions. The reason is that we want to carefully select N as a function of SNR, a problem that was left open in [12]. Note that we can not directly use [14] as the sum of two PAMs is not necessarily another PAM.

In multi-user settings, we may wish to select one user's input as Gaussian, another as discrete, or both mixtures of discrete and Gaussian. To handle such scenarios based on (5), we need bounds on the cardinality and minimum distance of sums of discrete constellations. If X and Y are two sets, denote the *sum-set* as $X+Y := \{x+y | x\in X, y\in Y\}$. Tight bounds on the cardinality and the minimum distance of $X+Y$, for general X and Y, are an open problem in number theory. The following set of sufficient conditions will play an important role in evaluating (3) under mixed inputs.

Proposition 1. *Let $(h_x,h_y)\in\mathbb{R}^2$ be two constants, $X \sim PAM(|X|, d_{\min(X)})$ and $Y \sim PAM(|Y|, d_{\min(Y)})$. Then $d_{\min(h_xX+h_yY)} = \min(|h_x|d_{\min(X)}, |h_y|d_{\min(Y)})$ and $|h_xX+h_yY| = |X||Y|$*

$$\textit{if } \; |Y||h_y|d_{\min(Y)} \le |h_x|d_{\min(X)} \tag{6}$$

$$\textit{or if } \; |X||h_x|d_{\min(X)} \le |h_y|d_{\min(Y)}, \tag{7}$$

Proof: The conditions in (6)-(7) are such that one PAM constellation is completely contained within two points of the other PAM constellation. ∎

When Proposition 1 is not applicable, we will use the following proposition inspired by [15]:

Proposition 2. *Let X and Y be two $PAM(N,d_{\min})$. Then $|h_xX+h_yX| = |X|^2$ almost everywhere and*

$$d_{\min(h_xX+h_yY)} \ge \min\left(|h_x|,|h_y|\right)\min\left(1,\gamma\right)d_{\min} \tag{8}$$

for all $(h_x,h_y)\in E\subseteq\mathbb{R}^2$ where the complement of E has Lesgebue measure smaller than 2γ, for any $\gamma>0$.

Proof: The proof can be found in Appendix A. ∎

III. Achievable Sum-rate with Mixed Inputs

We now evaluate the lower bound in (3) with inputs

$$X_i = \sqrt{1-\delta}\; X_{iD} + \sqrt{\delta}\; X_{iG}, \quad \delta\in[0,1], \tag{9a}$$

$$X_{iD} \sim \mathrm{PAM}\left(N, \sqrt{\frac{12}{N^2-1}}\right),\; X_{iG}\sim\mathcal{N}(0,1), \tag{9b}$$

where X_{ij} are independent for $i\in[1:2], j\in\{D,G\}$. The input in (9) has two parameters: the number of points N (since here we consider unitary power constraint), and the power split

TABLE I: Parameters used in the proof of Theorems 2 and 3.

α	N	δ	$d^2_{\min(\mathcal{S})}$	$d_L(\alpha)$	Additive Gap
$[0, 1/2]$	Not Applicable	1	Not Applicable	$2(1-\alpha)$	1 bit [10], for $\mathsf{S} \geq \mathsf{I}(1+\mathsf{I})$
$(1/2, 2/3]$	$\left\lfloor \sqrt{1+\left(\frac{\mathsf{I}^2}{1+\mathsf{S}+2\mathsf{I}}\right)^{1-\epsilon}} \right\rfloor$	$\frac{1}{1+\mathsf{I}}$	$\frac{\mathsf{I}^2}{1+\mathsf{S}+2\mathsf{I}} \frac{12}{N^2-1}$	$(2-\epsilon)\alpha$	eq.(15), for $\mathsf{S} < \mathsf{I}(1+\mathsf{I})$, $\mathsf{I}^3 \leq \mathsf{S}(1+\mathsf{S}+2\mathsf{I})$
$(2/3, 2)$	$\left\lfloor \sqrt{1+\sqrt{\mathsf{I}}} \right\rfloor$	$\frac{1}{1+\mathsf{I}}$	$\frac{\mathsf{I}\min(\mathsf{S},\mathsf{I})}{1+\mathsf{S}+2\mathsf{I}} \frac{12}{N^2-1} \min(1,\gamma^2)$	$\max(\alpha, 2-\alpha)$ except on an outage set of measure $\leq 2\gamma$	eq.(16), for $\mathsf{I}^3 > \mathsf{S}(1+\mathsf{S}+2\mathsf{I})$, $\mathsf{I} < \mathsf{S}(1+\mathsf{S})$
$[2, \infty)$	$\left\lfloor \sqrt{1+\mathsf{S}^{1-\epsilon}} \right\rfloor$	0	$\mathsf{S}\frac{12}{N^2-1}$	$2(1-\epsilon)$	eq.(14) for $\mathsf{I} \geq \mathsf{S}(1+\mathsf{S})$

δ. Careful choices of these parameters will lead to the desired results in different regimes. The inputs in (9) are symmetric, i.e., same parameters for both users, since we restrict attention to a symmetric IC. Extensions to a non-symmetric ICs are straightforward but more computationally involved.

Proposition 3. *For the input in* (9)*, $R_L(\mathsf{S},\mathsf{I})$ may be further lower bounded as*

$$R_L(\mathsf{S},\mathsf{I}) \geq 2\mathsf{I_d}\left(|\mathcal{S}|, \frac{d^2_{\min(\mathcal{S})}}{4}\right) + \log\left(1+\frac{\mathsf{S}\delta}{1+\mathsf{I}\delta}\right) - \min\left(\log(N^2), \log\left(1+\frac{\mathsf{I}(1-\delta)}{1+\mathsf{I}\delta}\right)\right),$$

where the sum-set $\mathcal{S}$ is defined as

$$\mathcal{S} := \left\{ \frac{\sqrt{1-\delta}}{\sqrt{1+\mathsf{S}\delta+\mathsf{I}\delta}}(\sqrt{\mathsf{S}}x_{1D}+\sqrt{\mathsf{I}}x_{2D}) : \begin{array}{l} x_{1D} \in X_{1D} \\ x_{2D} \in X_{2D} \end{array} \right\},$$

Proof: Due to the symmetry of the problem $I(X_1;Y_1) = I(X_2;Y_2) = R_L(\mathsf{S},\mathsf{I})/2$; hence, for a $Z \sim \mathcal{N}(0,1)$, we have

$$\begin{aligned} I(X_2;Y_2) &= h(\sqrt{\mathsf{I}}X_1+\sqrt{\mathsf{S}}X_2+Z) - h(\sqrt{\mathsf{I}}X_1+Z) \\ &= \underbrace{\left(h\left(\frac{\sqrt{1-\delta}}{\sqrt{1+\mathsf{S}\delta+\mathsf{I}\delta}}(\sqrt{\mathsf{S}}x_{1D}+\sqrt{\mathsf{I}}x_{2D})+Z\right)-h(Z)\right)}_{\geq \mathsf{I_d}\left(|\mathcal{S}|, \frac{d^2_{\min(\mathcal{S})}}{4}\right) \text{ by Theorem 1}} \\ &- \underbrace{\left(h\left(\frac{\sqrt{1-\delta}}{\sqrt{1+\mathsf{I}\delta}}\sqrt{\mathsf{I}}X_{1D}+Z\right)-h(Z)\right)}_{=I\left(\frac{\sqrt{1-\delta}}{\sqrt{1+\mathsf{I}\delta}}\sqrt{\mathsf{I}}X_{1D}+Z;X_{1D}\right) \leq \min\left(\log(N), \frac{1}{2}\log(1+\frac{\mathsf{I}(1-\delta)}{1+\mathsf{I}\delta})\right)} \\ &+ \frac{1}{2}\log(1+\mathsf{I}\delta+\mathsf{S}\delta) - \frac{1}{2}\log(1+\mathsf{I}\delta), \end{aligned}$$

where the mutual information upper bound follows since 'Gaussian maximizes the differential entropy for a given second moment constraint' and 'a uniform input maximizes the entropy of a discrete random variable' [1]. ■

IV. High SNR performance

In this Section we show that the lower bound in Proposition 3 attains the same generalized degrees of freedom (gDoF) as the upper bound in (4), where gDoF is defined as

$$d_L(\alpha) := \lim_{\mathsf{S}\to\infty} \frac{R_L(\mathsf{S},\mathsf{I}=\mathsf{S}^\alpha)}{\frac{1}{2}\log(1+\mathsf{S})}. \tag{10}$$

Similarly, define $d_U(\alpha)$ by replacing R_L by R_U in (10). With the upper bound in (4) we have [10]

$$d_U(\alpha) = 2\min\left(1, \max\left(\frac{\alpha}{2}, 1-\frac{\alpha}{2}\right), \max(\alpha, 1-\alpha)\right). \tag{11}$$

The main result of this Section is:

Theorem 2. *A mixed input as in Proposition 3 achieves the gDoF summarized in Table I.*

Proof: The parameters of the input in (9) are chosen as in Table I. When $\alpha \leq \frac{1}{2}$, Gaussian inputs are optimal to within 1 bit [10], hence we set $\delta = 1$. For $\alpha \in (1/2, 2)$, we choose δ such that the interference from the Gaussian portion of the input, which is treated as noise, is 'below the level of the noise' [10] by setting $\delta = \frac{1}{1+\mathsf{I}}$. For $\alpha \geq 2$, we choose to send only discrete inputs by setting $\delta = 0$.

In *very strong interference* $\mathsf{S}(1+\mathsf{S}) \leq \mathsf{I} \iff \alpha \geq 2$, we set $\delta = 0$ and $N = \lfloor\sqrt{1+\mathsf{S}^{1-\epsilon}}\rfloor$, for some $\epsilon \in (0,1)$ where $\epsilon < 1$ insures a non-vanishing rate as S increases and $\epsilon > 0$ insures that the minimum distance increases as S increases. With this δ and N, the condition in (6) is $N^2\mathsf{S} \leq \mathsf{I}$, which is readily verified in this regime. Therefore $|\mathcal{S}| = N^2$ and $\frac{d^2_{\min(\mathcal{S})}}{4} = \frac{3\mathsf{S}}{N^2-1} \approx \mathsf{S}^\epsilon$ from Proposition 1. By plugging these values in Proposition 3, an achievable sum-rate is

$$\begin{aligned} R_L &\geq 2\mathsf{I_d}\left(N^2, \frac{3\mathsf{S}}{N^2-1}\right) - \min\left(\log(N^2), \log(1+\mathsf{I})\right) \\ &\geq \log(N^2) - \log\left(\frac{\mathrm{e}}{2}\right) - 2\log\left(1+(N^2-1)\mathrm{e}^{-\frac{3\mathsf{S}}{N^2-1}}\right). \end{aligned} \tag{12}$$

Finally, in the high-SNR limit we obtain $\lim_{\mathsf{S}\to\infty} \frac{\text{eq.(12)}}{0.5\log(1+\mathsf{S})} = 2(1-\epsilon)$. The *moderately weak interference* regime $\alpha \in (1/2, 2/3]$ follows similarly: now the condition in Proposition 1 is $N^2\mathsf{I} \leq \mathsf{S}$ and the sum-rate is given by (13) (see next) because $\delta = \frac{1}{1+\mathsf{I}}$.

For the *remaining regime* $\alpha \in (2/3, 2)$, we set $N = \left\lfloor\sqrt{1+\sqrt{\mathsf{I}}}\right\rfloor \approx \mathsf{S}^{\alpha/4}$ and $\delta = \frac{1}{1+\mathsf{I}}$ and therefore, from Proposition 2, $|\mathcal{S}| = N^2$ and $d^2_{\min(\mathcal{S})}$ as in Table I almost surely for all channel parameters. Here $d^2_{\min(\mathcal{S})}$ increases in S (i.e., no need for the ϵ parameter as in the previous regimes) but the result holds for all channel parameters up to an outage set of measure less than 2γ, where $\gamma > 0$ also affects the minimum distance. By plugging these values in Proposition 3 we can further lower bound the achievable sum-rate as

$$R_L \geq \log(N^2) - \log\left(\frac{\mathrm{e}}{2}\right) - 2\log\left(1+(N^2-1)\mathrm{e}^{-\frac{d^2_{\min(\mathcal{S})}}{4}}\right) + \log\left(1+\frac{\mathsf{S}}{1+2\mathsf{I}}\right). \tag{13}$$

Finally, in the high-SNR limit we obtain $\lim_{\mathsf{S}\to\infty}\frac{\text{eq.(13)}}{0.5\log(1+\mathsf{S})} = \alpha + 2[1-\alpha]^+ = \max(2-\alpha,\alpha)$. ■

Theorem 2 shows that a mixed input can achieve the optimal (same as the classical IC) gDoF: exactly in very weak interference $\alpha \in [0, 1/2]$, arbitrarily close in moderately weak interference $\alpha \in (1/2, 2/3]$ and very strong interference $\alpha > 2$, and up to an outage set, similar to [15], in $\alpha \in (2/3, 2]$. This result, especially for the strong and very strong interference regimes, is unexpected. In the classical IC, which we use as an upper bound, we know that i.i.d. Gaussian inputs with *joint decoding* of the intended and interfering message is optimal in strong and very strong interference. The lower bound $R_L(\mathsf{S},\mathsf{I})$ however seems to imply TIN; if this interpretation of the mutual information expression of $R_L(\mathsf{S},\mathsf{I})$ were correct, then we would conclude that 0 gDoF are achievable if both users are active because the strong interference 'swamps' the useful signal at each receiver and since it is Gaussian, it is the "worst" find of noise; indeed, i.i.d. Gaussian inputs give $R_L(\mathsf{S},\mathsf{I}) = \log\left(1+\frac{\mathsf{S}}{1+\mathsf{I}}\right)$ that corresponds to 0 gDoF for $\mathsf{I} \geq \mathsf{S}$. Our work shows that by selecting non-Gaussian inputs with more structure, and by exploiting knowledge of this structure leads to unbounded gains (gDoF gains) even when interference is treated as noise.

V. Finite SNR performance

In this Section we 'refine' the gDoF result of Theorem 2 by proving an additive gap between a sum-capacity upper bound in (4) and the sum-rate achievable with a mixed input in Proposition 3. This improves the result of Section IV and demonstrates that the classical IC gDoF are exactly achievable.

Theorem 3. *A mixed input as in Proposition 3 achieves the gap summarized in Table I.*

Proof: We analyze the different regimes separately. We only consider the case $\mathsf{S} \geq 1$, otherwise a trivial gap of 1 bit can be achieved by silencing both users, and $\mathsf{I} \geq 1$, otherwise a trivial gap of 1 bit can be achieved by using Gaussian inputs and treating interference as noise, i.e., $\delta = 1$ in (9).

Very strong interference $\alpha \geq 2$: in (12), in addition to setting the parameters as in Table I, we further set $\epsilon = \left[\frac{\log(\frac{1}{3}\ln(\mathsf{S}))}{\log(\mathsf{S})}\right]^+$ so that $\log\left(1+(N^2-1)\mathrm{e}^{-\frac{3\mathsf{S}}{N^2-1}}\right) \leq 1$ bit. The gap is the difference between R_U in (4a) and R_L and is bounded as

$$\begin{aligned}
R_U - R_L &\leq \log(1+\mathsf{S}) - \log\left(\lfloor\sqrt{1+\mathsf{S}^{1-\epsilon}}\rfloor^2\right) + \log\left(\frac{\mathrm{e}}{2}\right) + 2 \\
&\overset{(a)}{\leq} \log(1+\mathsf{S}) - 2\log\left(\frac{1}{2}\sqrt{1+\mathsf{S}^{1-\epsilon}}\right) + \log\left(\frac{\mathrm{e}}{2}\right) + 2 \\
&\overset{(b)}{\leq} \log(\mathsf{S}^\epsilon) + \log\left(\frac{\mathrm{e}}{2}\right) + 4 \\
&\overset{(c)}{\leq} \left[\log\left(\frac{1}{3}\ln(\mathsf{S})\right)\right]^+ + \log\left(\frac{\mathrm{e}}{2}\right) + 4,
\end{aligned} \tag{14}$$

where: (a) $\lfloor x\rfloor \geq \frac{1}{2}x$ for $x \geq 1$, (b) $\frac{1+x}{1+x^{1-\epsilon}} \leq x^\epsilon$ for $x \geq 1$, and (c) definition of ϵ.

Moderately weak interference regime $1/2 < \alpha \leq 2/3$: the gap analysis is similar to the very strong interference regime but we now use the upper bound in (4b): we first notice that the difference between the upper bound in (4b) and the last term in (13) can be upper bounded as

$$\begin{aligned}
&\log\left(1+\mathsf{I}+\frac{\mathsf{S}}{1+\mathsf{I}}\right) - \log\left(1+\frac{\mathsf{S}}{1+2\mathsf{I}}\right) \\
&\leq \log\left(1+\frac{\mathsf{I}^2}{1+\mathsf{S}+2\mathsf{I}}\right) + \log(2)
\end{aligned}$$

therefore, the gap analysis is the same as that leading to (14) if we replace S with $\frac{\mathsf{I}^2}{1+\mathsf{S}+2\mathsf{I}}$ and we add an extra bit; we therefore conclude that in this regime the gap is

$$\begin{aligned}
R_U - R_L &\leq \left[\log\left(\frac{1}{3}\ln\left(\frac{\mathsf{I}^2}{1+\mathsf{S}+2\mathsf{I}}\right)\right)\right]^+ + \log\left(\frac{\mathrm{e}}{2}\right) + 5 \\
&\leq \left[\log\left(\frac{1}{3}\ln\left(\frac{\mathsf{S}}{3}\right)\right)\right]^+ + \log\left(\frac{\mathrm{e}}{2}\right) + 5,
\end{aligned} \tag{15}$$

where the last inequality follows since $\mathsf{I} \leq \mathsf{S}$ in this regime.

Very weak interference $\alpha \in [0, 1/2]$: Gaussian inputs and treat interference as noise are optimal to within 1 bit [10].

Remaining regime $\alpha \in (2/3, 2)$: the difference between the upper bound in (4c) and the last term in (13) satisfies

$$\begin{aligned}
&\frac{1}{2}\log^+\left(\frac{1+\mathsf{S}}{1+\mathsf{I}}\right) + \frac{1}{2}\log(1+\mathsf{I}+\mathsf{S}) - \log\left(1+\frac{\mathsf{S}}{1+2\mathsf{I}}\right) \\
&\leq \frac{1}{2}\log(1+\mathsf{I}) + \log(2)
\end{aligned}$$

thus the gap, with $N = \left\lfloor\sqrt{1+\sqrt{\mathsf{I}}}\right\rfloor$, is

$$\begin{aligned}
R_U - R_L \leq &\underbrace{\frac{1}{2}\log(1+\mathsf{I}) - \log(N^2)}_{\leq 2 \text{ for } \mathsf{I}\geq 0} + 1 + \log\left(\frac{\mathrm{e}}{2}\right) \\
&+ 2\log\left(1+(N^2-1)\mathrm{e}^{-\frac{d^2_{\min(\mathcal{S})}}{4}}\right),
\end{aligned}$$

where the last term is not bounded for $\alpha = 2/3$ or $\alpha = 2$ because at these points $d_{\min}$ does not increase with S. We remedy this by slightly changing the number of points of the discrete part of the input to $N = \lfloor\sqrt{1+(\sqrt{\mathsf{I}})^{1-\epsilon}}\rfloor$ and pick $\epsilon = \left[\frac{2}{\log(\mathsf{I})}\log\left(\frac{2\sqrt{2}\ln(\mathsf{I})}{3\min(1,\gamma^2)}\right)\right]^+$ to ensures that $\log(1+(N^2-1)\exp(-d^2_{\min(\mathcal{S})}/4)) \leq 1$. With this we can show

$$\begin{aligned}
R_U - R_L &\leq \frac{1}{2}\log(1+\mathsf{I}) - \log\left(\lfloor\sqrt{1+\sqrt{\mathsf{I}}^{1-\epsilon}}\rfloor^2\right) + \log(4\mathrm{e}) \\
&\leq \left[\log\left(\frac{2\sqrt{2}\ln(\sqrt{\mathsf{I}})}{3\min(1,\gamma^2)}\right)\right]^+ + 3.5 + \log\left(\frac{\mathrm{e}}{2}\right).
\end{aligned} \tag{16}$$

We note however, in strong interference for $\frac{4\sqrt{\mathsf{I}}\ln(\sqrt{\mathsf{I}})}{3\min(1,\gamma^2)} \le \mathsf{S}$, and in moderate interference for $\frac{\mathsf{S}\ln(\mathsf{I})}{3\min(1,\gamma^2)} \le \mathsf{I}^{\frac{3}{2}}$, we have that $\log(1+(N^2-1)\exp(-d^2_{\min(\mathcal{S})}/4)) \le 1$; therefore ϵ is not needed and the gap is $R_U - R_L \le 5 + \log\left(\frac{e}{2}\right)$. ■

VI. Applications of our results

We have evaluated a very simple, generally applicable lower bound to the capacity of any IC with a mixture of discrete and Gaussian inputs and shown that, through careful choice of the discrete input (where the number of points may depend on the SNR), that such an input may achieve to within an approximately constant gap of the capacity of a classical IC. This result is of interest in several channels, as outlined next.

In [5] the authors studied the block asynchronous IC, whose sum-capacity is denoted by $C^{\text{AC-IC}}$. It was shown in [5] that $R_L(\mathsf{S},\mathsf{I}) \le C^{\text{AC-IC}} \le R_U(\mathsf{S},\mathsf{I})$. Hence, our results apply directly and imply that using mixed inputs of the type proposed here we may approximately (in the sense of Theorem 3) achieve the capacity of the classical, synchronous IC.

Another application is to the IC with oblivious receivers introduced in [4], where both receivers lack knowledge of the codebook of the interfering transmitter. Lack of codebook knowledge of the interfering signal prevents the decoders from using joint decoding of messages and successive interference cancellation, which are known to be capacity achieving in a classical IC (where all codebooks are available to all nodes). Denote the sum-capacity of the oblivious IC by $C^{\text{OB-IC}}$. [4] demonstrated that $R_L(\mathsf{S},\mathsf{I}) = C^{\text{OB-IC}}$ is indeed the capacity of the oblivious IC, however, the input distribution that maximizes $R_L(\mathsf{S},\mathsf{I})$ was not found. Our results again directly apply and demonstrate the surprising fact that, even if receivers do not know the interfering codebooks and cannot perform joint decoding of messages, using a mixture of discrete and Gaussian inputs, one can overcome this lack of codebook knowledge and "approximately" achieve the capacity of the IC where all codebooks are known.

VII. Conclusion

We studied the performance of mixed inputs on the Gaussian IC. Its application to oblivious and asynchronous ICs somewhat surprisingly implies that much less "global coordination" between nodes is needed than one might expect: synchronism and codebook knowledge might not be critical if one is happy with "approximate" capacity results. We showed that TIN is gDoF optimal and within $O(\log\log(\mathsf{S}))$ of capacity.

Appendix

Let $\mathcal{S} = h_x X + h_y Y$ and $z_* \in \mathbb{Z}$. We want to find $d_{\min(\mathcal{S})} := \min_{i\neq j}\{|s_i - s_j| : s_i, s_j \in \mathcal{S}\}$ with $|s_i - s_j| = |h_x x_i + h_y y_i - h_x x_j - h_y y_j|$.

Case 1) $x_i = x_j$ and $y_i \neq y_j$, or $x_i \neq x_j$ and $y_i = y_j$:

$$|s_i - s_j| = |h_y| d_{\min}(Y)\ |z_i - z_j| \ge |h_y| d_{\min}(Y), \text{ or}$$
$$|s_i - s_j| \ge |h_x| d_{\min}(X).$$

Case 2) $x_i \neq x_j$ and $y_i \neq y_j$:

$$|s_i - s_j| = |h_y d_{\min}(Y)| \left| \frac{h_x d_{\min}(X)}{h_y d_{\min}(Y)}(z_{xi} - z_{xj}) - (z_{yj} - z_{yi}) \right|.$$

Next, we lower bound $|hz_1 - z_2|$ where $h := \frac{h_x d_{\min}(X)}{h_y d_{\min}(Y)}$. Let $E = \{h : |hz_1 - z_2| \ge \gamma\}$ for some $\gamma > 0$. The Lebesque measure of E^c (this will be our outage set) is

$$m(E^c) = m(\{h : \frac{-\gamma + z_2}{z_1} < h < \frac{\gamma + z_2}{z_1}\}) = \frac{2\gamma}{|z_1|} < 2\gamma.$$

Hence, on a set E we have the following lower bound $|s_i - s_j| \ge |h_y d_{\min}(Y)|\gamma$. By symmetry, $|s_i - s_j| \ge |h_x d_{\min}(X)|\gamma$. Putting all the cases together we have:

$$d_{\min(\mathcal{S})} \ge \min\left(|h_y| d_{\min}(Y), |h_x| d_{\min}(X)\right) \min(1,\gamma).$$

Now, assume $|X| = |Y| = N$ and $d_{\min(Y)} = d_{\min(X)}$. If $|S| \neq N^2$, then there exists $s_i = h_x x_i + h_y y_i$ and $s_j = h_x x_j + h_y y_j$, such that $s_i = s_j$ for some $i \neq j$, which in turn implies that $\exists x_i, y_i, x_j, y_j$ such that

$$\frac{h_x}{h_y} = \frac{y_j - y_i}{x_i - x_j} = \frac{|z_{yi} - z_{yj}| d_{\min}(Y)}{|z_{xi} - z_{xj}| d_{\min}(X)} = \frac{|z_{yi} - z_{yj}|}{|z_{xi} - z_{xj}|}, \quad (17)$$

i.e., $\frac{h_x}{h_y} \in \mathbb{Q}$ in order for $|S| \neq N^2$. Since, rationals have measure zero this implies that $|S| = N^2$ a.e..

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