Two-Way AWGN Channel Error Exponents at Zero Rate

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***Abstract*—Achievable error exponent regions of a two-way additive white Gaussian noise (AWGN) channel, where two terminals exchange a fixed number of messages M, are derived. In particular, error exponent regions for $M = 2$ messages under expected power and $M = 3$ messages under almost sure power constraints are considered. For $M = 2$ messages the use of active feedback is shown to lead to an error exponent gain over that when feedback / interaction is ignored. For $M = 3$ messages and asymmetric channels, it is shown that the error exponent of the weaker channel may be improved through active feedback, at the expense of a decreased error exponent of the stronger direction. This may, for sufficiently asymmetric channel gains, outperform the error exponent region achieved by having both terminals operate independently of one another (ignoring the possibility of sending feedback for the other).**

I. Introduction

Error exponents characterize the rate of decay of the probability of error with blocklength as a function of the channel parameters and number of messages (or rate). Here, we focus for the first time on the two-dimensional error exponent region of the two-way additive white Gaussian noise channel

$$Y_1 = X_1 + X_2 + N_1,\ N_1 \sim \mathcal{N}(0, \sigma_1^2) \quad (1)$$

$$Y_2 = X_1 + X_2 + N_2,\ N_2 \sim \mathcal{N}(0, \sigma_2^2) \quad (2)$$

where for $i = 1, 2$, $X_i \in \mathbb{R}$ are the channel inputs, $Y_i \in \mathbb{R}$ are the channel outputs and N_i are independent Gaussian noises each independent and identically distributed. In a two-way AWGN channel two terminals exchange messages through a shared channel, and channel inputs at time i may be a function not only of the messages, but also of the past channel outputs available at that terminal (termed interaction, or adaptation).

When the additive Gaussian noises are independent and memoryless, the two-dimensional capacity region decomposes into two parallel one-way, interference-free channels, forming a rectangular capacity region. Interaction, or adapting current channel inputs to past channel outputs, does *not* increase the capacity region. Similarly, in the one-way memoryless AWGN channel, feedback (even perfect) does not increase the capacity region. However, it is known that perfect feedback can dramatically increase the error exponents of the one-way channel [1]–[3], and that even noisy feedback may do so in some scenarios [4]–[7] (to be described in detail later). In a two-way setting, data and potential "feedback" share the same time, frequency, and power resources over noisy channels. The central question is whether interaction may improve the two-way error exponent region over that where the two directions act as two one-way channels (no feedback is sent). We address this problem at zero-rate, where $M = 2$ and $M = 3$ messages are transmitted by each user. A full version of this paper with proofs in the Appendix is accessible at: https://www.ece.uic.edu/bin/view/Devroye/Publications

A. Past work: AWGN error exponents at zero rate

Shannon [8] demonstrated that one can transmit at positive rates R while guaranteeing that the probability of error $P_e(n)$ tends to zero as the blocklength $n \to \infty$, and defined the reliability, or error exponent, for a channel at rate R as:

$$E(R) = \lim_{n\to\infty} -\frac{1}{n} \log \mathsf{P}_e(n) \quad (3)$$

where $\mathsf{P}_e(n)$ is the smallest probability of error that can be achieved by a code of rate R with blocklength n. Let $X_{i,k}$ denote the input of terminal i at time k. Two types of power constraints are commonly considered:

- Almost sure (AS) power constraint: $\sum_{k=1}^{n} X_{i,k}^2 \leq nP$,
- Expected (EXP) power constraint: $E\left[\sum_{k=1}^{n} X_{i,k}^2\right] \leq nP$.

To the best of our knowledge, error exponents for the two-way AWGN channel have not been studied. Most related is the work on error exponents of the one-way AWGN channel at zero rate; we omit the large body of work on one-way AWGN channels with feedback at positive rates, see for example [9], [10] and the references therein.

For the transmission of M messages, let $E_{\text{FB}}\left(M, \frac{P}{\sigma^2}, \frac{P_{\text{FB}}}{\sigma^2_{\text{FB}}}\right)$ denote the error exponent of a one-way channel with noisy feedback (FB) and forward and back channel SNRs of $\frac{P}{\sigma^2}$, and $\frac{P_{\text{FB}}}{\sigma^2_{\text{FB}}}$. We drop the third argument and the subscript for channels without FB, and use the super-scripts AS, EXP to denote the power constraint. Shannon [11] demonstrated achievability of the following exponent for AS power constraints for a one-way channel without FB using a simplex code:

$$E^{\text{AS}}\left(M, \frac{P}{\sigma^2}\right) = \frac{P}{\sigma^2}\frac{M}{4(M-1)}. \quad (4)$$

While it is known that feedback cannot improve the capacity of memoryless channels, Pinsker [3] showed that the use of noiseless feedback ($\sigma^2_{\text{FB}} \to 0$) improves the error exponent for any fixed $M \geq 3$ up to $\frac{P}{2\sigma^2}$. This is not the case for $M = 2$,

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since for binary transmission no improvements are possible for an AS power constraint even with perfect feedback [12].

More relevant to the study of two-way AWGN channels are one-way AWGN channels with noisy feedback, as studied by Yamamoto and Burnashev [7], [13]–[15] and Kim et al. [4]–[6] for $M = 2$ and $M = 3$, using active (receiver may encode information to be sent over the feedback link) and passive (no encoding, simply sends the output) feedback.

1) Error exponents for $M = 2$: We outline results for active feedback, as this is most relevant to the two-way channel where re-encoding in the reverse link is natural (and may mimic passive). Kim et al. [5] showed for the channel in Fig. 1, that for an EXP power constraint in the forward, and either AS or EXP power constraints on the FB link, achievable and inachievable error exponent expressions are:

1) **AS power constraint**: $\sum_{k=1}^{n} g_k^2(Y^k) \leq nP_{\text{FB}}$

$$E_{\text{FB}}^{\text{AS}}\left(2, P, \sigma^2, P_{\text{FB}}, \sigma_{\text{FB}}^2\right) \geq \frac{P}{2\sigma^2} + \frac{2P_{\text{FB}}}{\sigma_{\text{FB}}^2} \tag{5}$$

$$E_{\text{FB}}^{\text{AS}}\left(2, P, \sigma^2, P_{\text{FB}}, \sigma_{\text{FB}}^2\right) \leq \frac{P}{2\sigma^2} + \frac{2\sqrt{(P_{\text{FB}} + \sigma_{\text{FB}}^2)P_{\text{FB}}}}{\sigma_{\text{FB}}^2} \tag{6}$$

2) **EXP power constraint**: $\mathsf{E}\left[\sum_{k=1}^{n} g_k^2(Y^k)\right] \leq nP_{\text{FB}}$

$$E_{\text{FB}}^{\text{EXP}}\left(2, P, \sigma^2, P_{\text{FB}}, \sigma_{\text{FB}}^2\right) \geq \frac{2P}{\sigma^2} + \frac{2P_{\text{FB}}}{\sigma_{\text{FB}}^2} \tag{7}$$

$$E_{\text{FB}}^{\text{EXP}}\left(2, P, \sigma^2, P_{\text{FB}}, \sigma_{\text{FB}}^2\right) \leq \tag{8}$$
$$\frac{(\sqrt{P + \sigma^2} + \sqrt{P})^2}{\sigma^2} + \frac{(\sqrt{P_{\text{FB}} + \sigma_{\text{FB}}^2} + \sqrt{P_{\text{FB}}})^2}{\sigma_{\text{FB}}^2}$$

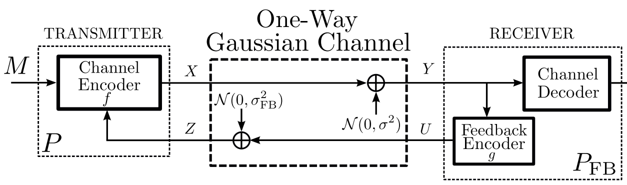

Fig. 1. One-way AWGN channel with Active Noisy Feedback

2) Error exponents for $M \geq 3$: Kim et al. [6] studied the reliability for $M \geq 3$ messages subject to the Peak Energy constraint (PE) on the forward channel, $\Pr\left\{\sum_{i=1}^{n} f_i^2(m, \tilde{Y}^{i-1}) \leq nP\right\} = 1, \forall m \in \{1, 2, ..., M\}$, and passive feedback, demonstrating:

$$E_{\text{FB}}^{\text{PE}}\left(M, \frac{P}{\sigma^2}, s\right) \geq \frac{P}{2\sigma^2}\left(1 - \frac{3(M-2)}{M(s^2 - 2s + 4) + 3(M-2)}\right), \tag{9}$$

where parameter $s \in [0, 1]$ characterizes the feedback noise variance $\sigma_{\text{FB}}^2(s)$. For $M = 3$ a sufficiently small $\sigma_{\text{FB}}^2(s) < \frac{1}{4}$, yields a gain over the non-feedback error exponent of Eq. (4).

II. Definitions

We study the two-way AWGN channel [16], [17] of Figure 2, where messages transmitted from the i-th terminal ($i \in \{1, 2\}$) are denoted by M_i, selected uniformly from $\mathcal{M}_i := \{1, 2, \cdots, M\}$, and transmitted to terminal $3 - i$. The channel model in (1)-(2) may be equivalently represented as

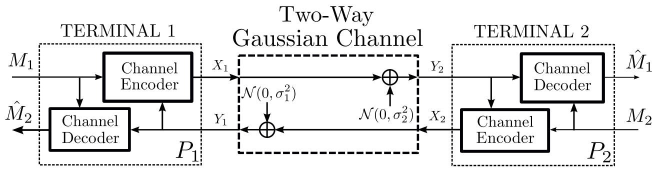

Fig. 2. Two-Way AWGN Channel Model.

$Y_2 = X_1 + N_2$, $Y_1 = X_2 + N_1$ since each user knows its own transmission and hence may subtract it. Let X_i^k denote the sequence of x_i variables of length k, where this length k should be clear from context.

A $\left(\frac{P_1}{\sigma_2^2}, \frac{P_2}{\sigma_1^2}, n\right)$ blocklength-n code consists of two encoding and two decoding rules. Each terminal's encoding rule comprises a set of n functions $x_{i,k} : \{1, 2, ..., M\} \times Y_i^{k-1} \to \mathcal{X}_i, \quad k = 1, ...n$, leading to the k-th channel inputs $X_{i,k} = x_{i,k}\left(M_i, Y_i^{k-1}\right)$, subject to either EXP or AS power constraints of P_i in each direction. The decoding rule for the i-th terminal ϕ_i estimates the received message based on Y_i^n as: $\phi_i : \mathcal{Y}_i^n \times \mathcal{M}_i \to \mathcal{M}_{3-i}$, $i = 1, 2$. Let $P_{e_{12}}^X\left(M, \frac{P_1}{\sigma_2^2}, \frac{P_2}{\sigma_1^2}, n\right) = \sum_{m_1, m_2} \Pr(\phi_1(y_1^n, m_1) \neq m_2 | m_1, m_2 \text{ sent})$ and similarly $P_{e_{21}}^X\left(M, \frac{P_1}{\sigma_2^2}, \frac{P_2}{\sigma_1^2}, n\right)$ denote the probability of error in the forward and backward directions simultaneously achieved by a particular $(\frac{P_1}{\sigma_2^2}, \frac{P_2}{\sigma_1^2}, n)$ code subject to $X \in \{\text{AS}, \text{EXP}\}$ power constraints for the two-way AWGN channel.

Definition 1. *A pair of error exponents $\left(E_{12}\left(M, \frac{P_1}{\sigma_2^2}, \frac{P_2}{\sigma_1^2}\right), E_{21}\left(M, \frac{P_1}{\sigma_2^2}, \frac{P_2}{\sigma_1^2}\right)\right)^X$ is called achievable at zero rate under X power constraint for the two-way AWGN channel, if there exists a $\left(\frac{P_1}{\sigma_2^2}, \frac{P_2}{\sigma_1^2}, n\right)$ code such that for large n, simultaneously*

$$-\frac{1}{n}\log P_{e_{12}}^X\left(M, \frac{P_1}{\sigma_2^2}, \frac{P_2}{\sigma_1^2}\right) > E_{12}\left(M, \frac{P_1}{\sigma_2^2}, \frac{P_2}{\sigma_1^2}\right), \text{ and} \tag{10}$$

$$-\frac{1}{n}\log P_{e_{21}}^X\left(M, \frac{P_1}{\sigma_2^2}, \frac{P_2}{\sigma_1^2}\right) > E_{21}\left(M, \frac{P_1}{\sigma_2^2}, \frac{P_2}{\sigma_1^2}\right). \tag{11}$$

Definition 2. *The error exponent region for the two-way AWGN channel at zero rate corresponds to the union over all achievable error exponent pairs $(E_{12}, E_{21})^X$, where we will often drop the arguments of E_{ij} for notational simplicity.*

III. Contributions

The study of two-way error exponents highlights the tension between the dedication of resources to one terminal's own transmission versus the interactive encoding of the message and received signals to possibly aid in the transmission of the other terminal's message (through, for example, feedback). Improving the other user's error exponent may come at the price of reducing one's own. This setting differs from one-way settings where all resources are exclusively used to communicate one message. Our results – an initial characterization of two-way error exponents at zero-rate – rely on carefully tailored schemes for one-way communications with feedback. We summarize the main propositions next, some are immediately obvious, and further details are provided in the Appendix of the longer online version.

One simple achievability scheme for both transmitters to act as two one-way channels, with no power allocated to FB.

Proposition 1. *Under both AS and EXP power constraints the following error exponents are achievable*

$$E_{12}\left(M, \frac{P_1}{\sigma_2^2}, \frac{P_2}{\sigma_1^2}\right) \geq \frac{P_1}{\sigma_2^2}\frac{M}{4(M-1)}, \tag{12}$$

$$E_{21}\left(M, \frac{P_1}{\sigma_2^2}, \frac{P_2}{\sigma_1^2}\right) \geq \frac{P_2}{\sigma_1^2}\frac{M}{4(M-1)}. \tag{13}$$

This proposition follows directly from Shannon [11].

A simple upper bound for the AS power constraint is obtained by providing perfect output feedback information to both transmitters (like two one-way channels with perfect feedback). This yields the outer bound region below.

Proposition 2. *The error exponent region for AS power constraints is bounded by:*

$$E_{12}^{AS}\left(M, \frac{P_1}{\sigma_2^2}, \frac{P_2}{\sigma_1^2}\right) \leq \frac{P_1}{2\sigma_2^2}, \; E_{21}^{AS}\left(M, \frac{P_1}{\sigma_2^2}, \frac{P_2}{\sigma_1^2}\right) \leq \frac{P_2}{2\sigma_1^2}.$$

We conclude from Proposition 1 and 2, that the error exponent region for $M=2$ subject to an AS power constraint corresponds *exactly* to the one obtained by non-feedback transmission in each direction. For the EXP power constraint, Proposition 1 and the following Proposition 3, in which Kim et al's achievability scheme [5] for a one-way noisy FB channel is time-shared in the two directions, are the best error exponent regions the authors have been able to derive to date.

Proposition 3. *An achievable error exponent region for $M=2$ under EXP power constraints is the union over all error exponent pairs $(E_{12}, E_{21})^{EXP}$ over $\lambda \in [0,1]$ for which:*

$$E_{12}^{EXP}\left(M=2, \frac{P_1}{\sigma_2^2}, \frac{P_2}{\sigma_1^2}\right) \geq \lambda K_1 \frac{2P_1}{\sigma_2^2} + (1-\lambda) J_2 \frac{2P_2}{\sigma_1^2} \tag{14}$$

$$E_{21}^{EXP}\left(M=2, \frac{P_1}{\sigma_2^2}, \frac{P_2}{\sigma_1^2}\right) \geq \lambda K_2 \frac{2P_2}{\sigma_1^2} + (1-\lambda) J_1 \frac{2P_1}{\sigma_2^2} \tag{15}$$

for $K_1, K_2 \in [0, \frac{1}{\lambda}]$ and $J_1, J_2 \in [0, \frac{1}{1-\lambda}]$ such that $\lambda K_i + (1-\lambda) J_i \leq 1$ for $i = 1, 2$.

For $M=3$ messages we consider asymmetric $\frac{P_1}{\sigma_2^2} < \frac{P_2}{\sigma_1^2}$ as we have been unable to yield gains over Proposition 1 for symmetric SNRs. Noisy feedback appears to need to be much less noisy than the direct link to provide gains [6].

Proposition 4. *An achievable error exponent region for $M=3$ under an AS power constraint and using **passive feedback** is the union over all error exponent pairs $(E_{12}, E_{21})^{AS}$ over $\lambda, \lambda_1, s \in [0,1]$ satisfying:*

$$E_{21}^{AS}\left(M=3, \frac{P_1}{\sigma_2^2}, \frac{P_2}{\sigma_1^2}\right) \geq \frac{3}{8}\lambda \frac{P_2}{\sigma_1^2} \tag{16}$$

$$E_{12}^{AS}\left(M=3, \frac{P_1}{\sigma_2^2}, \frac{P_2}{\sigma_1^2}\right) \geq \tag{17}$$

$$\min\left\{\frac{P_1}{\sigma_2^2}\frac{\lambda_1(s^2-2s+4)}{8}, \frac{P_2}{\sigma_1^2}\frac{3s^2\lambda_1(1-\lambda)}{32}, \frac{P_1}{\sigma_2}\frac{\left(1-\frac{\lambda_1}{4}\right)}{2}\right\}.$$

The last expression may be reduced by equating terms 1 and 3, leaving λ_1 as a function of s as $\lambda_1 = \frac{4}{s^2-2s+5}$, yielding:

$$E_{12}^{\text{AS}}\left(M=3, \frac{P_1}{\sigma_2^2}, \frac{P_2}{\sigma_1^2}\right) \geq \tag{18}$$

$$\min\left\{\frac{P_1}{\sigma_2^2}\left(\frac{s^2-2s+4}{2(s^2-2s+5)}\right), \frac{P_2}{\sigma_1^2}\frac{3s^2(1-\lambda)}{8(s^2-2s+5)}\right\}.$$

The above follows from using a non-feedback transmission for M_2 in λn channel uses, and [6] (Section II-B) for M_1 in the remaining $(1-\lambda)n$ channel uses.

Proposition 5. *An achievable error exponent region for $M=3$ under an AS power constraint and using **active feedback** is the union over all error exponent pairs $(E_{12}, E_{21})^{AS}$ over all $\lambda \in [0,1]$ and $s \in [0,1]$ satisfying:*

$$E_{21}^{AS}\left(M=3, \frac{P_1}{\sigma_2^2}, \frac{P_2}{\sigma_1^2}\right) \geq \frac{3}{8}\lambda \frac{P_2}{\sigma_1^2} \tag{19}$$

$$E_{12}^{AS}\left(M=3, \frac{P_1}{\sigma_2^2}, \frac{P_2}{\sigma_1^2}\right) \geq$$

$$\min\left\{\frac{P_1}{\sigma_2^2}\left(\frac{1}{2}\frac{s^2-2s+4}{s^2-2s+5}\right), \frac{3}{8}(1-\lambda)\frac{P_2}{\sigma_1^2}\right\}. \tag{20}$$

An inachievability outer bound for $M=2$ under EXP power constraints follows directly from using the outer bound for the one-way channel using active feedback of [5] in each direction, as in (8). This appears to be loose.

Proposition 6. *An outer bound on the error exponent region for $M=2$ under EXP power constraint corresponds the union over all error exponent pairs that satisfy:*

$$E_{12}^{EXP}\left(M=2, P_1, \sigma_2^2, P_2, \sigma_1^2\right) \leq \tag{21}$$

$$\frac{\left(\sqrt{P_1+\sigma_2^2}+\sqrt{P_1}\right)^2}{\sigma_2^2} + \frac{\left(\sqrt{P_2+\sigma_1^2}+\sqrt{P_2}\right)^2}{\sigma_1^2}$$

$$E_{21}^{EXP}\left(M=2, P_1, \sigma_2^2, P_2, \sigma_1^2\right) \leq \tag{22}$$

$$\frac{\left(\sqrt{P_2+\sigma_1^2}+\sqrt{P_2}\right)^2}{\sigma_1^2} + \frac{\left(\sqrt{P_1+\sigma_2^2}+\sqrt{P_1}\right)^2}{\sigma_2^2}.$$

IV. Sketch of Main Result: Proposition 5

For the channel in Figure 2, with $\frac{P_2}{\sigma_1^2} > \frac{P_1}{\sigma_2^2}$, an AS power constraint, consider a two-phase transmission scheme of n channel uses split as in Figure 3, where $\lambda, \lambda_1, \lambda_2 \in [0,1]$, and $\lambda_3 = 1 - \lambda_1 - \lambda_2$.

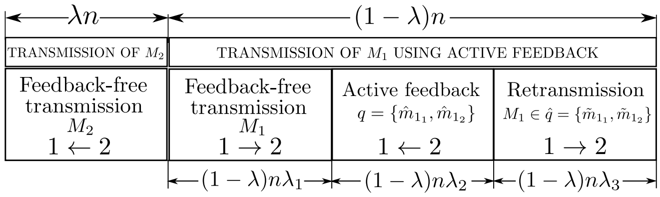

Fig. 3. Time-sharing approach for the transmission of M_1 and M_2.

Then, we define the simplex code $\mathcal{C}(\Omega, E_\omega)$ for the transmission of three symbols $\omega \in \Omega := \{\omega_1, \omega_2, \omega_3\}$ from the i-th transmitter with stage duration j, using symbol energy E_ω and as the code with codewords $X_i^j(\omega)$ of length j:

$$X_i^j(w) = \begin{cases} \sqrt{E_\omega}\cdot(0,1,0,...,0), & \text{if } \omega=\omega_1 \\ \sqrt{E_\omega}\cdot(\frac{-\sqrt{3}}{2},-\frac{1}{2},0,...,0), & \text{if } \omega=\omega_2 \\ \sqrt{E_\omega}\cdot(\frac{+\sqrt{3}}{2},-\frac{1}{2},0,...,0), & \text{if } \omega=\omega_3 \end{cases}. \quad (23)$$

For the first λn channel uses, terminal 2 uses code $\mathcal{C}(\mathcal{M}_2, \lambda n P_2)$ to transmit $m_2 \in \mathcal{M}_2 = \{1,2,3\}$ as codewords $X_2^{\lambda n}(m_2)$ over the stronger link, yielding $E_{21}\left(M, \frac{P_1}{\sigma_2^2}, \frac{P_2}{\sigma_1^2}\right) = \frac{3}{8}\lambda\frac{P_2}{\sigma_1^2}$, from (4). The remaining power of $(1-\lambda)nP_2$ is used to provide feedback for the transmission of M_1. The latter involves $(1-\lambda)n$ channel uses over three stages: transmission, active feedback, and retransmission, as in Figure 3.

Transmission: During $(1-\lambda)n\lambda_1$ channel uses, message m_1 is transmitted as codeword $X_1^{\lambda_1 n}(m_1)$. The encoder of terminal 1 uses the simplex code $\mathcal{C}(\mathcal{M}_1, \lambda_1 n P_1)$. The signal received at terminal 2 is decoded using a protection region for each transmitted codeword (see Fig. 5 in [6]), parametrized by $s\in[0,1]$. A transmission error in this stage occurs if $y_2^{\lambda_1 n}$ is received in the wrong protection region or between the two wrong codewords 2 and 3 (that is within A'_{23}; WLOG assume $m_1 = 1$ is sent), then let this error event be denoted as $\mathcal{E}_T$. The achievable error exponent of this stage is derived from $\mathsf{P}(\mathcal{E}_T)$ as in Eq. 7 of [6] as:

$$\lim_{n\to\infty} -\frac{1}{n}\log\left(\mathsf{P}(\mathcal{E}_T)\right) := E_{\mathcal{E}_T} \geq \frac{\lambda_1 P_1}{8\sigma_2^2}(s^2-2s+4). \quad (24)$$

Active feedback: If the received signal is not within a protection region then terminal 2 determines that the true message could be one out of two possible candidates, namely, one of the most likely codeword pair $q=\{\hat{m}_{1_1}, \hat{m}_{1_2}\}$. Then, for $A'_{12}, A'_{13}, A'_{23}$ defined as in [6, Fig. 5] we set:

$$q = \begin{cases} \{1,2\}, & \text{if } y_2^{\lambda_1 n} \in A'_{12} \\ \{2,3\}, & \text{if } y_2^{\lambda_1 n} \in A'_{23} \\ \{1,3\}, & \text{if } y_2^{\lambda_1 n} \in A'_{13} \end{cases} \quad (25)$$

Let $\mathcal{Q}$ be the set of q values defined above. Then, this decision is sent back to terminal 1 encoded as $X_2^{\lambda_2 n}(q)$ using the simplex code $\mathcal{C}(\mathcal{Q}, (1-\lambda)nP_2)$ over the strongest link $1 \leftarrow 2$. Terminal 1 estimates q as $\hat{q} = \{\tilde{m}_{1_1}, \tilde{m}_{1_2}\}$, by means of the decoding regions of Fig. 4.

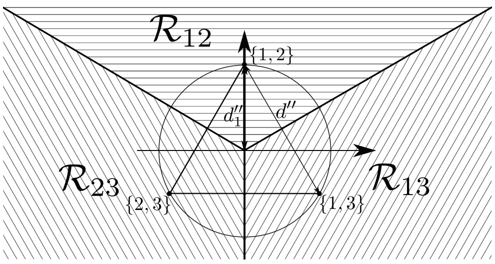

Fig. 4. Decoding regions for active feedback. Here, $d'' = \sqrt{3(1-\lambda)nP_2}$ and $d''_1 = \sqrt{(1-\lambda)nP_2}$.

The decoder errs if after this stage if $q \neq \hat{q}$, that is, assuming $M_1 = 1$ is sent, either of the following events occur, all given that the transmission stage yielded a signal received outside the protection regions:

$$\mathcal{E}_{\text{AFB}_1} = \left\{\left(y_1^{\lambda_2 n} \in \mathcal{R}_{13}\cup\mathcal{R}_{23}\right) \cap \left(y_2^{\lambda_1 n} \in A'_{12}\right)\right\}$$
$$\mathcal{E}_{\text{AFB}_2} = \left\{\left(y_1^{\lambda_2 n} \in \mathcal{R}_{12}\cup\mathcal{R}_{23}\right) \cap \left(y_2^{\lambda_1 n} \in A'_{13}\right)\right\}.$$

Let $\mathsf{P}(\mathcal{E}_{\text{AFB}})$ denote the probability that either of the two events happen. Then $\mathsf{P}(\mathcal{E}_{\text{AFB}}) \leq \mathsf{P}(\mathcal{E}_{\text{AFB}_1}) + \mathsf{P}(\mathcal{E}_{\text{AFB}_2}) = 2\mathsf{P}(\mathcal{E}_{\text{AFB}_1})$, and it can be shown that the corresponding error exponent is:

$$\lim_{n\to\infty} -\frac{1}{n}\log\left(\mathsf{P}(\mathcal{E}_{\text{AFB}})\right) := E_{\mathcal{E}_{\text{AFB}}} \geq \frac{3}{8}(1-\lambda)\frac{P_2}{\sigma_1^2}. \quad (26)$$

Retransmission: At this point, both terminals have estimates of the two most likely codewords, $q=\{\hat{m}_{1_1}, \hat{m}_{1_2}\}$, and $\hat{q} = \{\tilde{m}_{1_1}, \tilde{m}_{1_2}\}$, which we assume are the same otherwise this is counted as an error in the feedback stage. Next, terminal 1 uses antipodal signaling to transmit the true codeword among these two candidates using the remaining power:

$$X_1^{\lambda_3 n} = \begin{cases} \sqrt{(1-\lambda_1)nP_1}\cdot(+1,0,...,0), & \text{if } m_1 = \min\{\tilde{m}_{1_1}, \tilde{m}_{1_2}\} \\ \sqrt{(1-\lambda_1)nP_1}\cdot(-1,0,...,0), & \text{if } m_1 = \max\{\tilde{m}_{1_1}, \tilde{m}_{1_2}\} \\ (0,0,...,0), & \text{otherwise.} \end{cases}$$

Decoding: Following [6], terminal 2 decodes M_1 immediately at the end of the *transmission stage* if $y_2^{\lambda_1 n}$ is received within a protection region. Otherwise, the decoder finds the most likely codeword pair and sends it back to terminal 1. After the feedback stage, terminal 2 waits for the retransmission signal which is used along with the result of the transmission stage to determine $\hat{m}_1$ using the decoding rule (27), as in [6] for the case of a one-way channel with passive feedback:

$$\hat{m}_1 = \arg\min_{m_1\in\{\hat{m}_{1_1},\hat{m}_{1_2}\}} ||x_1^{(1-\lambda)n}(m_1) - y_2^{(1-\lambda)n}|| \quad (27)$$
$$= \arg\min_{m_1\in\{\hat{m}_{1_1},\hat{m}_{1_2}\}} \left(||x_1^{\lambda_1 n}(m_1) - y_2^{\lambda_1 n}||^2 + ||x_1^{\lambda_3 n}(m_1) - y_2^{\lambda_3 n}||^2\right)^{\frac{1}{2}}. \quad (28)$$

The decoder errs in this stage if, given that the transmission stage led to a signal outside the protection regions and the feedback led to $q=\hat{q}$, the following event occurs:

$$\mathcal{E}_{\text{RT}} = \{m_1 \in q = \{\hat{m}_{1_1}, \hat{m}_{1_2}\} \cap m_1 \in \hat{q} = \{\tilde{m}_{1_1}, \tilde{m}_{1_2}\} \cap \hat{m}_1 \neq m_1\}. \quad (29)$$

Then, the error exponent (30) can be derived in a similar way as $\mathsf{P}(\mathcal{E}_2)$ in Section II-A of [6] yielding:

$$\lim_{n\to\infty} -\frac{1}{n}\log\left(\mathsf{P}(\mathcal{E}_{\text{RT}})\right) := E_{\mathcal{E}_{\text{RT}}} \geq \left(1-\frac{\lambda_1}{4}\right)\frac{P_1}{2\sigma_2}. \quad (30)$$

The overall error exponent expression for the $1 \to 2$ direction is derived from Equations (24), (26) and (30), as the minimum of the three stages' decays, for parameter $s\in[0,1]$:

$$E_{12}\left(M, \frac{P_1}{\sigma_2^2}, \frac{P_2}{\sigma_1^2}\right) \geq \min\left\{\frac{\lambda_1 P_1}{8\sigma_2^2}(s^2-2s+4), \frac{3}{8}(1-\lambda)\frac{P_2}{\sigma_1^2}, \left(1-\frac{\lambda_1}{4}\right)\frac{P_1}{2\sigma_2}\right\}. \quad (31)$$

Equating the first and third terms, related to terminal's 1 transmissions, allows expressing λ_1 as a function of s: $\lambda_1 = \frac{4}{s^2-2s+5}$. This reduces the number of argumens of (31) to two, which leads to (20). Note that if $\frac{P_2}{\sigma_1^2} >> \frac{P_1}{\sigma_2^2}$, (20) is dominated by the first term, and consequently, determined by the choice $s, \lambda \in [0,1]$ (size of protection region and amount of power allocated to the $1 \leftarrow 2$ link). The maximum error exponent for the $1 \rightarrow 2$ direction is thus $\frac{2}{5}\frac{P_1}{\sigma_2^2}$, yielding a gain of 6.67% gain over the $\frac{3}{8}\frac{P_1}{\sigma_2^2}$ achieved without interaction/feedback. This gain results from a choice of $s = 0$. However, from (20), we see that in general, any choice of $s \in (0,1)$ leads to an error exponent gain in the $1 \rightarrow 2$ direction over the feedback-free transmission if $\lambda \leq 1 - \frac{16}{15}\left(\frac{\text{SNR}_1}{\text{SNR}_2}\right)$.

V. Error exponent region numerical comparisons

Figure 5(a) shows achievable error exponent regions for the two-way AWGN channel for the transmission of two messages. The solid line denotes the achievable region using active feedback under the EXP power constraint (Proposition 3) and the dotted line (lower left) the achievable region for independent transmissions in both directions without feedback and AS power constraint (Proposition 1) which coincides with Proposition 2. Note how the active noisy feedback greatly improves the achievable region. The dashed line region corresponds to the loose outer bounds of Proposition 6.

Error exponent regions for $M = 3$ are shown in Fig. 5(b). The region of Proposition 1 is shown by the solid line rectangle (independent feedback-free transmission for each direction). The region of Proposition 5 is shown with a dashed line. Note the benefit of active feedback for the $1 \rightarrow 2$ direction over the error exponent ($E_{12} = 0.75$) attained by feedback-free transmissions in both directions.This comes at the expense of a reduction in the error exponent in the $1 \leftarrow 2$ direction. For the evaluated SNR pair the highest exponent in the $1 \rightarrow 2$ direction is attained at $E_{12} = 0.8$. Given the SNR ratio $\frac{\text{SNR}_2}{\text{SNR}_1} = 8$ used for Fig. 5(b), one can characterize the values of λ for which pairs of the form $(E_{12} = 0.8, E_{21})^{AS}$ are achievable, as for $0 \leq \lambda \leq 13/15$. For $13/15 < \lambda \leq 1$, no gain in E_{12} is possible even though terminal 2 allocates power (and hence decreases its own E_{21}) to provide feedback to terminal 1. Moreover, a dotted line depicts Proposition 4. Note that for the evaluated SNRs, passive feedback is unable to provide a gain for the $1 \rightarrow 2$ direction, event though terminal's 2 error exponent is significantly exacerbated.

When $\frac{\text{SNR}_2}{\text{SNR}_1} \rightarrow \infty$, then $E_{12} \rightarrow \infty$. The other direction however, can be seen as a one-way channel with perfect feedback, for which $E_{12} \rightarrow \frac{\text{SNR}_1}{2}$ (or $E_{12} \leq 1$ in Figure 5(b)).

VI. Conclusion

Error exponents for the two-way AWGN channel are for the first time investigated. Our findings for $M = 2$ under EXP constraints and general SNRs and for $M = 3$ under AS constraints in non-symmetric SNR settings, show that feedback/interaction can be exploited towards improving the error exponent region (unlike the capacity result, in which the two directions decouple entirely). For the asymmetric setting, the error exponent of the weaker link may be increased at the cost of a small decrease on the error exponent of the stronger link. Ongoing work seeks to extend results for $M = 2$ under EXP constraints to $M = 3$.

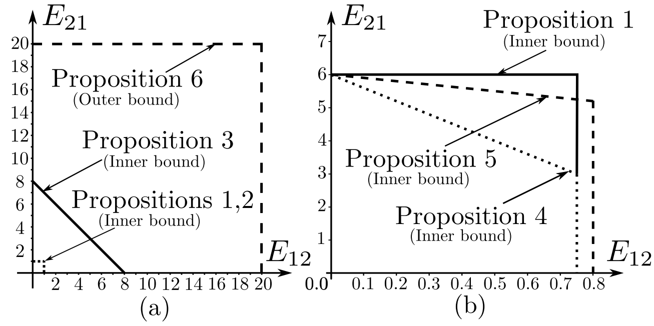

Fig. 5. Comparisons of error exponent regions for $M = 2$ and $M = 3$.

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Appendix

A. Probability of error analysis for the achievability scheme for $M = 3$ under AS power constraint and non-symmetric SNR.

As indicated in Section IV, messages M_1 and M_2 are transmitted in different stages. The first λn channel uses are used for transmitting M_2, using a non-feedback transmission over the strongest link. Transmitter at terminal 2 takes advantage of the high SNR by sending its own message while saving some power to help the other direction once M_2 is transmitted. The transmission of M_1 is performed in the remaining $(1-\lambda)n$ channel uses over three stages: transmission, active feedback and retransmission. This communication scheme is based on extending the approach proposed in [6], with the remarkable difference of using encoded feedback instead of passive feedback. Equation (20) from Proposition 5 represents an achievable error exponent in the $1 \to 2$ direction as the minimum between error exponents of the different stages, coupled by s and λ parameters. During each stage's description, we provide some definitions that are required for the upcoming derivations. Moreover, for illustration purposes, consider Figure 6 and the simplex code $\mathcal{C}(\mathcal{M}_1, \lambda_1 nP_1)$ for the transmission stage, and Figure 4 and the simplex code $\mathcal{C}(\mathcal{Q}, (1-\lambda)nP_2)$ for the active feedback stage.

Note first that the transmission of M_1 lasts for $(1-\lambda)n$ channel uses, and that terminal 1 is required to transmit only during transmission and retransmission stages. Therefore we use λ_1 as a power splitting parameter that determines the fraction of power used for each stage, $\lambda_1 P_1$ and $(1-\lambda_1)P_1$ respectively. This is not the case of terminal 2, since it utilizes all the remaining power available after the transmission of M_2 to provide feedback during the transmission of M_1. It is understood that minimum distance decoding is used for all our derivations.

Definitions for transmission stage.
Decoding of the signal transmitted in this stage is realized by means of a protection region (as it is proposed in [6]) established for each transmitted codeword. These regions are defined in Equation (32) for a parameter $t \in [0, \frac{\sqrt{3}-1}{2}]$, geometrically linked to parameter $s \in [0,1]$, as indicated for the conditions stated in [6], and depicted in Figure 6. Each B_{m_1} region defines a safety zone for the corresponding codeword $m_1 \in \{1,2,3\}$, such that all signals received within a protection region are immediately declared as the respective codeword linked to such protection region.

$$\begin{aligned} B_{m_1} = \Big\{ y_2^{\lambda_1 n} : &||x_1^{\lambda_1 n}(m_1) - y_2^{\lambda_1 n}|| \leq ||x_1^{\lambda_1 n}(m_1') - y_2^{\lambda_1 n}|| \,, \\ &\text{for } m_1' \neq m_1 \\ &\Big| ||x_1^{\lambda_1 n}(m_1') - y_2^{\lambda_1 n}|| - ||x_1^{\lambda_1 n}(m_1'') - y_2^{\lambda_1 n}|| \Big| \leq td', \\ &\text{for } m_1', m_1'' \neq m_1 \Big\} \end{aligned} \tag{32}$$

Figure 6 illustrates regions $A'_{12}, A'_{13}, A'_{23}$ as well. These regions are different from those called *protection regions* in the sense that all the signals they receive in may be associated with two distinct codewords, following their definition in [6]. Assuming WLOG the transmission of $M_1 = 1$, we observe that the probability of error in this stage is linked to the the error event we denote by $\mathcal{E}_T$ (presented in Equation (34)) which represents a signal received at terminal 2 that is in the wrong protection region ($B_2 \cup B_3$) or within an ambiguous region that does not include $M_1 = 1$, i.e. the received signal lies in A'_{23}.

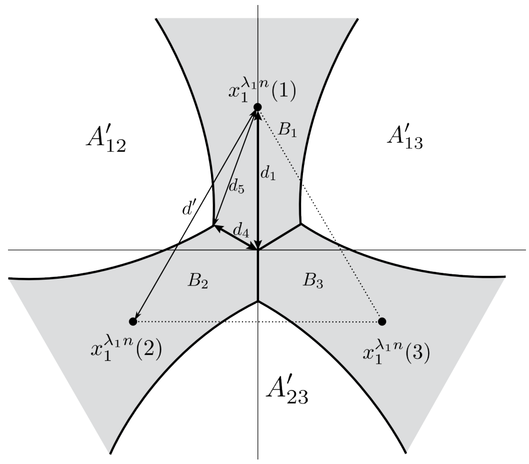

Fig. 6. Protection regions defined in [6] and used for the M_1 transmission stage. Here, $d_1 = \sqrt{\lambda_1 nP_1}$, $d_4 = \frac{s}{2}d_1$, $d' = \sqrt{3\lambda_1 nP_1}$, and $d_5 = \sqrt{\frac{\lambda_1 nP_1}{4\sigma_2^2}(s^2 - 2s + 4)}$.

Definitions for the active feedback stage:
Given the transmission of $M_1 = 1$, and in the event that terminal 2 received the signal outside the protection regions, i.e. $y_2^{\lambda_1 n} \in A'_{12} \cup A'_{13}$, our communication scheme makes terminal 2 to take a hard decoding decision about the received signal, by means of sending back to terminal 1 the most likely codeword pair q. Terminal 2 uses the simplex code $\mathcal{C}(\mathcal{Q}, (1-\lambda)nP_2)$ for this transmission, which is decoded at terminal's 1 end by using the decoding regions depicted in Figure 4, defined as:

$$\begin{aligned} \mathcal{R}_{m_1, m_1'} = \Big\{ y_1^{\lambda_2 n} : &||x_2^{\lambda_2 n}(q = \{m_1, m_1'\}) - y_1^{\lambda_2 n}|| \\ &\leq ||x_2^{\lambda_2 n}(q = \{m_1, m_1''\}) - y_1^{\lambda_2 n}|| \\ &\text{and,} \\ &||x_2^{\lambda_2 n}(q = \{m_1, m_1'\}) - y_1^{\lambda_2 n}|| \\ &\leq ||x_2^{\lambda_2 n}(q = \{m_1', m_1''\}) - y_1^{\lambda_2 n}|| \Big\}. \end{aligned} \tag{33}$$

Probability of error analysis.
As indicated in Section IV, our main result for the $1 \to 2$ direction is derived from Equations (24), (26) and (30), as the minimal decay among three arguments as described by Equation (31).

To prove our result, consider the communication scheme for the transmission of message $M_1 = 1$ we introduced in Section IV. Lets denote the probability of error for this transmission as $P^{12}(M_1 \neq \hat{M}_1 \mid M_1 = 1) = P^{12}_{M_1=1}(M_1 \neq \hat{M}_1)$ for notational convenience, assuming that $M_1 = 1$ is transmitted, and that $P^{12}_{M_1=1}(M_1 \neq \hat{M}_1) = P^{12}_{M_1=2}(M_1 \neq \hat{M}_1) = P^{12}_{M_1=3}(M_1 \neq \hat{M}_1)$. Then, we can express the probability of error for this transmission (recalling that all symbols in $\mathcal{M}_1$ are equally likely) as:

$$
\begin{aligned}
&P^{12}_1(M_1 \neq \hat{M}_1)\\
&= \mathsf{P}(\text{Err.Tx})\\
&+ \mathsf{P}\left(\text{Err.FB} \mid \overbrace{Y_2^{\lambda_1 n} \notin \{B_1 \cup B_2 \cup B_3\}}^{\text{Not in Prot.Reg.}}\right)\\
&\cdot \mathsf{P}(Y_2^{\lambda_1 n} \notin \{B_1 \cup B_2 \cup B_3\})\\
&+ \mathsf{P}\left(\text{Err.Rtx} \mid \overbrace{Y_2^{\lambda_1 n} \notin \{B_1 \cup B_2 \cup B_3\}}^{\text{Not in Prot.Reg.}} \cap \overbrace{\{M_1 \in q\} \cap \{M_1 \in \hat{q}\}}^{\text{No mis-coord. error}}\right)\\
&\cdot \mathsf{P}\left(\left\{Y_2^{\lambda_1 n} \notin \{B_1 \cup B_2 \cup B_3\}\right\} \cap \{M_1 \in q\} \cap \{M_1 \in \hat{q}\}\right),
\end{aligned}
$$

Where, Err.Tx, Err.FB and Err. Rtx correspond to error events for each corresponding stage described as:

- Error event in transmission stage:
$$\text{Err.Tx} := Y_2^{\lambda_1 n} \in \{B_2 \cup B_3 \cup A'_{23}\}.$$
- Error events in FB stage, subject that $Y_2^{\lambda_1 n}$ was not received within a protection region:
$$
\begin{aligned}
\text{Err.FB} := &\left\{\left(Y_1^{\lambda_2 n} \in \mathcal{R}_{12} \cup \mathcal{R}_{23}\right) \cap \left(Y_2^{\lambda_1 n} \in A'_{13}\right)\right\}\\
&\cup \left\{\left(Y_1^{\lambda_2 n} \in \mathcal{R}_{13} \cup \mathcal{R}_{23}\right) \cap \left(Y_2^{\lambda_1 n} \in A'_{12}\right)\right\}.
\end{aligned}
$$
- Error event in retransmission stage:
$$\text{Err.Rtx} := \{m_1 \neq \hat{m}_1\}.$$

Then,

$$
\begin{aligned}
&P^{12}_1(M_1 \neq \hat{M}_1)\\
&= \mathsf{P}(Y_2^{\lambda_1 n} \in \{B_2 \cup B_3 \cup A'_{23}\})\\
&+ \mathsf{P}\left(\left\{\left(Y_1^{\lambda_2 n} \in \mathcal{R}_{12} \cup \mathcal{R}_{23}\right) \cap \left(Y_2^{\lambda_1 n} \in A'_{13}\right)\right\}\right.\\
&\left.\cup \left\{\left(Y_1^{\lambda_2 n} \in \mathcal{R}_{13} \cup \mathcal{R}_{23}\right) \cap \left(Y_2^{\lambda_1 n} \in A'_{12}\right)\right\}\right)\\
&\cdot \mathsf{P}\left(Y_2^{\lambda_1 n} \notin \{B_1 \cup B_2 \cup B_3\}\right)\\
&+ \mathsf{P}\left(\{m_1 \neq \hat{m}_1\} \cap \left\{Y_2^{\lambda_1 n} \in \{A'_{12} \cup A'_{13} \cup A'_{23}\}\right\}\right.\\
&\left.\cap \{M_1 \in q\} \cap \{M_1 \in \hat{q}\}\right)
\end{aligned}
$$

$$
\begin{aligned}
&P^{12}_1(M_1 \neq \hat{M}_1)\\
&\leq \mathsf{P}(Y_2^{\lambda_1 n} \in \{B_2 \cup B_3 \cup A'_{23}\})\\
&+ \mathsf{P}\left(\left\{\left(Y_1^{\lambda_2 n} \in \mathcal{R}_{12} \cap \mathcal{R}_{23}\right) \cap \left(Y_2^{\lambda_2 n} \in A'_{13}\right)\right\}\right)\\
&\cdot \mathsf{P}\left(Y_2^{\lambda_1 n} \notin \{B_1 \cup B_2 \cup B_3\}\right)\\
&+ \mathsf{P}\left(\left\{\left(Y_1^{\lambda_2 n} \in \mathcal{R}_{13} \cap \mathcal{R}_{23}\right) \cap \left(Y_2^{\lambda_2 n} \in A'_{12}\right)\right\}\right)\\
&\cdot \mathsf{P}\left(Y_2^{\lambda_1 n} \notin \{B_1 \cup B_2 \cup B_3\}\right)\\
&+ P\left(\{m_1 \neq \hat{m}_1\} \cap \left\{Y_2^{\lambda_1 n} \in \{A'_{12} \cup A'_{13}\}\right\}\right.\\
&\left.\cap \{M_1 \in q\} \cap \{M_1 \in \hat{q}\}\right)
\end{aligned}
$$

We can write this as:

$$
\begin{aligned}
&P^{12}_1(M_1 \neq \hat{M}_1)\\
&\leq \mathsf{P}(Y_2^{\lambda_1 n} \in \{B_2 \cup B_3 \cup A'_{23}\})\\
&+ \mathsf{P}\left(\left\{Y_1^{\lambda_2 n} \in \mathcal{R}_{12} \cap \mathcal{R}_{23}\right\} \cap \left\{Y_2^{\lambda_2 n} \in A'_{13}\right\}\right)\\
&+ \mathsf{P}\left(\left\{Y_1^{\lambda_2 n} \in \mathcal{R}_{13} \cap \mathcal{R}_{23}\right\} \cap \left\{Y_2^{\lambda_2 n} \in A'_{12}\right\}\right)\\
&+ \mathsf{P}\left(\{m_1 \neq \hat{m}_1\} \cap \{M_1 \in q\} \cap \{M_1 \in \hat{q}\}\right).
\end{aligned}
$$

Finally, we express the above terms as the probability of joint events:

$$
\begin{aligned}
P_e \leq\ &\mathsf{P}(Y_2^{\lambda_1 n} \in \{B_2 \cup B_3 \cup A'_{23}\})\\
&+ \mathsf{P}\left(\left\{Y_1^{\lambda_2 n} \in \mathcal{R}_{12} \cap \mathcal{R}_{23}\right\} \cap \left\{Y_2^{\lambda_2 n} \in A'_{13}\right\}\right)\\
&+ \mathsf{P}\left(\left\{Y_1^{\lambda_2 n} \in \mathcal{R}_{13} \cap \mathcal{R}_{23}\right\} \cap \left\{Y_2^{\lambda_2 n} \in A'_{12}\right\}\right)\\
&+ \mathsf{P}(\{m_1 \neq \hat{m}_1\} \cap \{M_1 \in q\} \cap \{M_1 \in \hat{q}\})
\end{aligned}
$$

We can now conclude that the terms of the above equation corresponds to the error events we defined for each stage:

- First term is $\mathcal{E}_T$
- Second and third terms correspond to $\mathcal{E}_{AFB}$
- Third term is related to $\mathcal{E}_{RT}$

We can now, analyze the probability of occurrence of each of these events as:

Probability of Error in the Transmission stage:
Given the error event:

$$\mathcal{E}_T = \{y_1^{\lambda_1 n} \in \{B_2 \cup B_3 \cup A'_{23}\}\} \tag{34}$$

We can follow the procedure and geometry described in [6] (see Figure 6 above), we can upper bound the probability of this event as:

$$
\begin{aligned}
\mathsf{P}(\mathcal{E}_T) &\leq 2Q\left(\frac{d_5}{\sigma_2}\right) \qquad (35)\\
&\leq \exp\left(-n\frac{\lambda_1 P_1}{8\sigma_2^2}(s^2 - 2s + 4)\right),
\end{aligned}
$$

from which we can derive a lower bound on the error exponent yielding Equation (24).

Probability of Error in the Active feedback stage:
Given that the original transmission of $M_1 = 1$ was not received in a protection region, the feedback transmission may cause a mis-coordination error ($q \neq \hat{q}$), whenever the events defined in Equations (36) and (37) occur.

$$\mathcal{E}_{\text{AFB}_1} = \left\{\left(y_1^{\lambda_2 n} \in \mathcal{R}_{13} \cup \mathcal{R}_{23}\right) \cap \left(y_2^{\lambda_1 n} \in A'_{12}\right)\right\} \tag{36}$$

$$\mathcal{E}_{\text{AFB}_2} = \left\{\left(y_1^{\lambda_2 n} \in \mathcal{R}_{12} \cup \mathcal{R}_{23}\right) \cap \left(y_2^{\lambda_1 n} \in A'_{13}\right)\right\}. \tag{37}$$

Le us now define $\mathsf{P}(\mathcal{E}_{\text{AFB}}) = \mathsf{P}(\mathcal{E}_{\text{AFB}_1} \text{ or } \mathcal{E}_{\text{AFB}_2})$, which can be upper bounded as:

$$\mathsf{P}(\mathcal{E}_{\text{AFB}}) \leq \mathsf{P}(\mathcal{E}_{\text{AFB}_1}) + \mathsf{P}(\mathcal{E}_{\text{AFB}_2}) = 2\mathsf{P}(\mathcal{E}_{\text{AFB}_1}). \quad (38)$$

Therefore, we can upper bound the $\mathsf{P}(\mathcal{E}_{\text{AFB}_1})$ probability to obtain our result of interest for the feedback stage as:

$$\begin{aligned}\mathsf{P}(\mathcal{E}_{\text{AFB}_1}) &= \mathsf{P}\left(Y_1^{\lambda_2 n} \in \mathcal{R}_{13} \cup \mathcal{R}_{23} | Y_2^{\lambda_1 n} \in A'_{12}\right) \quad (39)\\ &\cdot \mathsf{P}(Y_2^{\lambda_1 n} \in A'_{12})\\ &\leq \mathsf{P}\left(Y_1^{\lambda_2 n} \in \mathcal{R}_{13} \cup \mathcal{R}_{23} | Y_2^{\lambda_1 n} \in A'_{12}\right)\\ &\leq 2Q\left(\frac{d''}{2\sigma_1}\right) = 2Q\left(\frac{\sqrt{3(1-\lambda)nP_2}}{2\sigma_1}\right)\\ &\leq \exp\left(-\frac{3(1-\lambda)nP_2}{8\sigma_1^2}\right),\end{aligned}$$

which leads to the error exponent expression of Equation (26).

Probability of error for the Retransmission stage: The probability of error of this stage is linked to the occurrence of the event described by Equation (40)

$$\begin{aligned}\mathcal{E}_{\text{RT}} = \{&m_1 \in q = \{\hat{m}_{1_1}, \hat{m}_{1_2}\} \cap \quad (40)\\ &m_1 \in \hat{q} = \{\tilde{m}_{1_1}, \tilde{m}_{1_2}\} \cap \hat{m}_1 \neq m_1\}.\end{aligned}$$

The errors produced in the previous stages are captured by the corresponding error events defined above. Therefore, this transmission assumes that the two previous stages are correct, which is equivalent to the noiseless passive feedback case analyzed in [6], in which the received signal Y is transmitted back to the transmitter such that both, transmitter and receiver know exactly what is the most likely pair of codewords the receiver has determined (for the active feedback setting of this scheme, this is equivalent to $q = \hat{q}$). Thus, the probability of error expression from which to obtain the error exponent of this stage is derived in a similar manner and given by:

$$\mathsf{P}(\mathcal{E}_{\text{RT}}) = Q\left(\sqrt{\left(1 - \frac{\lambda_1}{4}\right)\frac{P_1}{\sigma_2}n}\right), \quad (41)$$

from which it is possible to obtain the corresponding error exponent shown in Equation (30).

B. Proof of Proposition 3: On the achievability scheme for $M = 2$ under EXP power constraint.

This section focus on the transmission of a binary alphabet ($M = 2$) over a two-way AWGN channel. Each terminal transmits message $m_i \in \mathcal{M}_i = \{1, 2\}$, for $i = 1, 2$. Results of Equations (14) and (15) are obtained based on an extension of the work presented in [5]. We assume WLOG that messages $M_i = 1$ are being sent in each direction.

1) A Two-way Communications Building Block: First we propose a two-way communications building block based on the one-way building block proposed in [5].This operational component is aimed to transmit a single bit using n channel uses, subject to the EXP power constraint in the forward and backward links. This two-way building block differs from the one-way version in the sense that the transmission in the two directions occurs simultaneously, that is following the channel model described by Equations (1) and (2). Then, each receiver may simply ignore its own transmission and operate as in two decoupled and independent directions. The probability of error given that bit $M_i = m_i$ is sent in each direction follows directly from that obtained for the one-way channel building block, that is, as in Eq. (138) of [5]. Terminal's 2 estimate of M_1 is denoted by M'_1, while M'_2 indicates the estimate of M_2 at terminal 1. Let us denote the probability of error on decoding transmitted messages M_i, conditioning on $M_i = m_i$ being sent by:

$$P_{m_1}^{12}(M_1 \neq M'_1) = \mathsf{P}(M_1 \neq M'_1 \mid M_1 = m_1) \quad (42)$$
$$P_{m_2}^{21}(M_2 \neq M'_2) = \mathsf{P}(M_2 \neq M'_2 \mid M_2 = m_2), \quad (43)$$

where the upper scripts indicate the corresponding communication direction and should be clear from the context onwards. Then, given equally likely messages to be transmitted, $\mathsf{P}(M_i = 1) = \mathsf{P}(M_i = 2)$, we can derive: $P_1^{12}(M_1 \neq M'_1) \leq \exp\left(-n2\frac{P_1}{\sigma_2^2}\right)$ and $P_1^{21}(M_2 \neq M'_2) \leq \exp\left(-n2\frac{P_2}{\sigma_1^2}\right)$, which lead to error exponent expressions:

$$E_{12_{\text{BB}}}^{\text{EXP}}\left(2, \frac{P_1}{\sigma_2^2}\right) \geq 2\frac{P_1}{\sigma_2^2} \quad (44)$$
$$E_{21_{\text{BB}}}^{\text{EXP}}\left(2, \frac{P_2}{\sigma_1^2}\right) \geq 2\frac{P_2}{\sigma_1^2}, \quad (45)$$

where the subscript BB stands for *building block*.

2) A three phases communication scheme for the two-way AWGN Channel based on the use of the building block: Similarly to the one-way approach of [5], we employ a scheme that makes use the building block previously defined comprising three stages: *Transmission phase*, *Echo phase* and *Retransmission phase*. We use a time split parameter $\lambda \in [0, 1]$ to establish each stage's duration correspondingly to: $\lambda(n-1)$, $(1-\lambda)(n-1)$ and 1 channel uses.

Transmission phase: For this stage, the two-way building block is used for the simultaneous transmission of messages M_1 and M_2 using $\lambda(n-1)$ channel uses. We allocate power $K_1 P_1$ for the $1 \to 2$ direction transmission, and power $K_2 P_2$ for the $2 \to 1$ direction. We use parameters $K_1, K_2 \in [0, 1/\lambda]$ to indicate the fraction of the available power that is allocated for the transmission phase in each direction. Then, assuming the transmission of $M_1 = 1$ and $M_2 = 1$ we obtain from the building block probability of error analysis:

$$P_1^{12}(M_1 \neq M'_1) \leq \exp\left(-\lambda n \frac{2K_1 P_1}{\sigma_2^2}\right) \quad (46)$$
$$P_1^{21}(M_2 \neq M'_2) \leq \exp\left(-\lambda n \frac{2K_2 P_2}{\sigma_1^2}\right) \quad (47)$$

Echo phase: In this stage, the estimation of the messages transmitted during transmission phase are sent back to the terminals that originated them. We parametrize the fraction of power allocated for this transmission by means of $J_1, J_2 \in [0, 1/(1-\lambda)]$ respectively for each terminal. In this phase, the two-way communications building block is used during $(1-\lambda)(n-1)$ channel uses to send both, estimate M'_1 back to terminal 1 using power J_2P_2 and estimate M_2' back to terminal 2 using power J_1P_1. At the end of this stage, each terminal has an estimate of the feedback messages, denoted correspondingly by M_1'' for M_1' and M_2'' for M_2'. Since these transmissions are performed using the basic two-way building block, the probability of error M_1' and M_2' transmission is determined by:

$$P_1^{12}(M_1' \neq M_1'') \leq \exp\left(-(1-\lambda)n\frac{2J_2P_2}{\sigma_1^2}\right) \tag{48}$$

$$P_1^{21}(M_2' \neq M_2'') \leq \exp\left(-(1-\lambda)n\frac{2J_1P_1}{\sigma_2^2}\right). \tag{49}$$

After the echo phase, the probability of a decoding error in both stages, transmission and echo phase, is given by their product due to independence:

$$\begin{aligned} &P_1^{12}(M_1''=1, M_1'=2) \qquad (50)\\ &\leq \exp\left(-\lambda n\frac{2K_1P_1}{\sigma_2^2}\right)\cdot\exp\left(-(1-\lambda)n\frac{2J_2P_2}{\sigma_1^2}\right)\\ &= \exp\left(-\lambda n\frac{2K_1P_1}{\sigma_2^2}-(1-\lambda)n\frac{2J_2P_2}{\sigma_1^2}\right)\\ &= \exp\left(-n\cdot\left(\lambda K_1\frac{2P_1}{\sigma_2^2}+(1-\lambda)J_2\frac{2P_2}{\sigma_1^2}\right)\right). \end{aligned}$$

Equivalently for the other direction,

$$\begin{aligned} &P_1^{21}(M_2''=1, M_2'=2) \qquad (51)\\ &\leq \exp\left(-\lambda n\frac{2K_2P_2}{\sigma_1^2}\right)\cdot\exp\left(-(1-\lambda)n\frac{2J_1P_1}{\sigma_2^2}\right)\\ &= \exp\left(-n\cdot\left(\lambda K_2\frac{2P_2}{\sigma_1^2}+(1-\lambda)J_1\frac{2P_1}{\sigma_2^2}\right)\right). \end{aligned}$$

Retransmission phase: In this stage each terminal compares the true message with the feedback received after the echo phase. If the compared messages are different, i.e. $M_1'' \neq M_1$ or $M_2'' \neq M_2$, a very high amplitude antipodal signal is used to correct this error event, which occurs with exponentially small probability, as shown in [5], otherwise, zero is retransmitted. Such retransmission signals occur for one channel use only. Since those events occur with exponentially small probability, the EXP power constraint is satisfied.

At the end of the three phases scheme, both terminals estimate messages M_1 and M_2 based on the last observed received signals at channel use n, respectively: Y_{2_n} and Y_{1_n}. These signals are later compared with a determined very large threshold Υ_i (chosen as indicated in [5]), which determines the following decoding rule at both terminals, shown next for terminal 2 only:

$$\hat{M}_1 = \begin{cases} 1, & \text{if } Y_{2_n} > \Upsilon_2\\ M_1', & \text{if } |Y_{2_n}| \leq \Upsilon_2\\ 2, & \text{if } Y_{2_n} < -\Upsilon_2 \end{cases} \tag{52}$$

Then, it can be shown that the probability of error of sending messages $M_i = 1$, follows a similar way as in the one-way case:

$$\begin{aligned} P_1^{12}(M_1 \neq \hat{M}_1) &= P_1^{12}(M_1'=2, M_1''=1) \qquad (53)\\ &\leq \exp\left(-n\cdot\left(\lambda K_1\frac{2P_1}{\sigma_2^2}+(1-\lambda)J_2\frac{2P_2}{\sigma_1^2}\right)\right), \end{aligned}$$

and analogously,

$$\begin{aligned} P_1^{21}(M_2 \neq \hat{M}_2) &= P_1^{21}(M_2'=2, M_2''=1) \qquad (54)\\ &\leq \exp\left(-n\cdot\left(\lambda K_2\frac{2P_2}{\sigma_1^2}+(1-\lambda)J_1\frac{2P_1}{\sigma_2^2}\right)\right). \end{aligned}$$

Therefore, the error exponent can be derived, for the $1 \to 2$ direction as

$$E_{12}^{\text{EXP}}\left(2, \frac{P_1}{\sigma_2^2}, \frac{P_2}{\sigma_1^2}\right) \geq \lambda K_1\frac{2P_1}{\sigma_2^2}+(1-\lambda)J_2\frac{2P_2}{\sigma_1^2}, \tag{55}$$

and for the $2 \to 1$ direction, as

$$E_{21}^{\text{EXP}}\left(2, \frac{P_1}{\sigma_2^2}, \frac{P_2}{\sigma_1^2}\right) \geq \lambda K_2\frac{2P_2}{\sigma_1^2}+(1-\lambda)J_1\frac{2P_1}{\sigma_2^2}. \tag{56}$$

We observe how the parametrization determined by $\lambda, K_1, K_2, J_1,$ and J_2 allows us to distribute the total power available at each terminal. Consider the case of terminal 1, which is equivalent to that of terminal 2. The power allocation of terminal 1 for n channel uses is:

$$\overbrace{\lambda(n-1)K_1P_1}^{\text{transmission phase}}+\overbrace{(1-\lambda)(n-1)J_1P_1}^{\text{echo phase}} \leq nP_1, \tag{57}$$

which can be written as:

$$\left[\lambda(n-1)K_1+(1-\lambda)(n-1)J_1\right]P_1 \leq nP_1. \tag{58}$$

Then, we note that for large n, Equation (58) and its equivalent for terminal 2 lead to:

$$\lambda K_1+(1-\lambda)J_1 \leq 1 \tag{59}$$

$$\lambda K_2+(1-\lambda)J_2 \leq 1, \tag{60}$$

for $\lambda \in [0,1]$ and $K_1, K_2 \in [0, 1/\lambda)$ and $J_1, J_2 \in [0, 1/(1-\lambda))$, leading to the conditions presented in Proposition 3.

A numerical evaluation on these results demonstrated that the error exponent region achieved by this scheme matches the one resulting by simply applying the one-way achievability scheme presented in [5] under the EXP power constraint for each communication direction in a time-sharing basis using a time-splitting parameter λ. This approach causes the probability of error in transmitting bits M_1 and M_2 as:

$$P_1(\hat{M}_1 \neq M_1)_{TS} \leq \exp\left(-\lambda n\left(\frac{2P_1}{\sigma_2^2}+\frac{2P_2}{\sigma_1^2}\right)\right), \tag{61}$$

$$P_1(\hat{M}_2 \neq M_2)_{TS} \leq \exp\left(-(1-\lambda)n\left(\frac{2P_1}{\sigma_2^2}+\frac{2P_2}{\sigma_1^2}\right)\right). \tag{62}$$

Then, achievable error exponents can be derived as:

$$E_{12_{TS}}^{\text{EXP}}\left(2, \frac{P_1}{\sigma_2^2}, \frac{P_2}{\sigma_1^2}\right) \geq \lambda \left(\frac{2P_1}{\sigma_2^2} + \frac{2P_2}{\sigma_1^2}\right), \tag{63}$$

$$E_{21_{TS}}^{\text{EXP}}\left(2, \frac{P_1}{\sigma_2^2}, \frac{P_2}{\sigma_1^2}\right) \geq (1-\lambda) \left(\frac{2P_1}{\sigma_2^2} + \frac{2P_2}{\sigma_1^2}\right). \tag{64}$$