A Lattice Compress-and-Forward Scheme

Yiwei Song and Natasha Devroye
Department of Electrical and Computer Engineering
University of Illinois at Chicago
ysong34, devroye@ uic.edu

***Abstract*—We present a nested lattice-code-based strategy that achieves the random-coding based Compress-and-Forward (CF) rate for the three node Gaussian relay channel. To do so, we first outline a lattice-based strategy for the $(X + Z_1, X + Z_2)$ Wyner-Ziv lossy source-coding with side-information problem in Gaussian noise, a re-interpretation of the nested lattice-code-based Gaussian Wyner-Ziv scheme presented by Zamir, Shamai, and Erez. We use the notation $(X + Z_1, X + Z_2)$ Wyner-Ziv to mean that the source is of the form $X + Z_1$ and the side-information at the receiver is of the form $X+Z_2$, for independent Gaussian X, Z_1 and Z_2. We use this $(X + Z_1, X + Z_2)$ Wyner-Ziv scheme to implement a "structured" or lattice-code-based CF scheme for the Gaussian relay channel which achieves the same rate as the Cover-El Gamal CF rate achieved by random Gaussian codebooks.**

I. Introduction

Lattice codes have been shown to perform as well as random codes for certain Gaussian channels, and to outperform random codes for specific Gaussian channels. As much is known about lattice codes and their performance in single path (point-to-point, multiple access and broadcast where information flows along one path) source and channel coding scenarios, in this paper we take the next step towards the goal of demonstrating that lattices may mimic random codes by considering a simple multiple path Gaussian network – the simple three user Gaussian relay channel in which information may flow from the source to the destination along two paths. In [1] it was shown that lattice codes may achieve the Gaussian Decode-and-Forward rate of [2] for the Gaussian relay channel. We now demonstrate that lattice codes may also be used to achieve the same rate as that achieved by random Gaussian codebooks in the Compress-and-Forward (CF) rate of [2], for this channel.

Scenarios in which lattice codes achieve the same rates as random codes. Lattice codes (and lattice decoding) have been shown to be capacity achieving in the Additive White Gaussian Noise (AWGN) point-to-point channel, using a unique decoding technique [3] exploiting a carefully chosen Minimum Mean Squared Error (MMSE) scaling coefficient, and recently, using an alternative list decoding technique [1]. Lattices codes may also be constructed that achieve the capacity of the Gaussian Multiple Access Channel (MAC) [4] and the Gaussian Broadcast Channel (BC) [5]. The latter exploited the fact that lattice codes may achieve the dirty-paper coding channel capacity [5] by mimicing random binning techniques in a structured manner. Recently, using a lattice list-decoding technique, nested lattice codes were shown to achieve the Gaussian random coding Decode-and-Forward rate in the Gaussian relay channel [1]. Though not the focus of this paper, we note that the structure provided by lattice codes may also be used to outperform that of random codes in specific channels, see [6], [4], [1], [7].

Lattice codes for binning. In networks with side-information, the concept of binning, which effectively allows the transmitters and receivers to properly exploit this side-information, is critical. The usage of lattices and structured codes for binning (as opposed to random binning as previously proposed) in various types of networks was considered in a comprehensive fashion in [5]. Of particular interest to the problem considered here is the nested lattice-coding approach of [5] to the Gaussian Wyner-Ziv coding problem. The Wyner-Ziv coding problem is that of lossy source coding with correlated side-information at the receiver. One example of a Gaussian Wyner-Ziv problem is one in which the Gaussian source to be compressed is of the form $X + Z$, and the side-information available at the reconstructing node is X, for Z independent of X and Gaussian, which we term the $(X+Z, X)$ Wyner-Ziv problem. A lattice-scheme is provided in [5] for the $(X + Z, X)$ Wyner-Ziv problem. We consider a lattice Wyner-Ziv coding scheme for the slightly altered $(X + Z_1, X + Z_2)$ channel model in which the source to be compressed is of the form $X + Z_1$ and the side-information is of the form $X + Z_2$, for independent, Gaussian X, Z_1 and Z_2, a re-interpretation of the scheme of [5] for the $(X + Z, X)$ model (whose generalization was mentioned in a footnote [5]). We include this lattice-based scheme for the $(X + Z_1, X + Z_2)$ model for completeness, and use it to construct a CF scheme based on nested lattice codes which recovers the same achievable rate as the achievable CF rate [2] using random Gaussian codebooks for the Gaussian relay channel.

A Compress-and-Forward (CF) rate for the Gaussian relay channel. Cover and El Gamal first proposed a CF scheme for the three user relay channel in [2]. In it, the relay does not decode the message (as it would in the Decode-and-Forward scheme) but instead compresses its received signal and forwards the compression index. The destination first recovers the compressed signal, using its direct-link side-information (the Wyner-Ziv problem), and then proceeds to decode the message from the recovered compressed signal. The CF scheme is generalized to arbitrary relay networks in the recently proposed "noisy network coding" scheme [8]. Armed with a lattice Wyner-Ziv scheme, we mimic every step of the Cover and El Gamal's CF scheme using lattice codes and will show that the same rate as that achieved by random Gaussian codebooks (which are not known to achieve the CF bound in general [9]), may be achieved in a structured manner.

Contribution and paper organization. The central contribution of this work is the application of a general lattice-coding based Wyner-Ziv scheme to the Gaussian three node relay channel. In particular, in Section II we first outline our notation and nested lattice coding preliminaries. In Section III we outline a nested lattice-code based scheme for a $(X+Z_1, X+Z_2)$ Wyner-Ziv problem in Theorem 1, which is used in Section IV's Theorem 2, to show that the rate achieved by random Gaussian codes in Cover and El Gamal's Compress-and-Forward scheme may be achieved using nested lattice codes. Finally, we conclude in Section V. Given the structure of lattice codes, this may constitute a more practical implementation of Wyner-Ziv coding (as already noted in [5]), of the CF scheme, and is an important first step towards a generic "structured" achievability scheme for networks.

II. Preliminaries a nested lattice codes

We first outline our notation and definitions for nested lattice codes for transmission over AWGN channels, following those of [5], [10]. We note that [11], [5], [3] and in particular [12] offer more thorough treatments, and defer the interested reader to those works for more details. An n-dimensional lattice Λ is a discrete subgroup of Euclidean space $\mathbb{R}^n$ (of vectors $\mathbf{x}$, though we will denote these without the bold font as x) with Euclidean norm $||\cdot||$ under vector addition. We may define

- The *nearest neighbor lattice quantizer* of Λ as $Q_\Lambda(x) = \arg\min_{\lambda\in\Lambda} ||x-\lambda||$;
- The $\texttt{mod}\ \Lambda$ operation as $x \bmod \Lambda := x - Q_\Lambda(x)$, hence $x = Q_\Lambda(x) + (x \mod \Lambda)$;
- The *fundamental region of* Λ as the set of all points closer to the origin than to any other lattice point $\mathcal{V}(\Lambda) := \{x : Q(x) = \mathbf{0}\}$ which is of volume $V := \text{Vol}(\mathcal{V}(\Lambda))$;
- The *second moment per dimension of a uniform distribution over* $\mathcal{V}$ as $\sigma^2(\Lambda) := \frac{1}{V}\cdot\frac{1}{n}\int_{\mathcal{V}} ||x||^2\, dx$;
- The *Crypto lemma* [13] which states that $(x+U) \mod \Lambda$ (where U is uniformly distributed over $\mathcal{V}$) is an independent random variable uniformly distributed over $\mathcal{V}$.

Standard definitions of *Poltyrev good* and *Rogers good* lattices are used [3], and by [14] we are guaranteed the existence of lattices which are both Polytrev and Rogers good.

The proposed schemes will be based on *nested lattice codes*. To define these, consider two lattices Λ and Λ_c such that $\Lambda \subseteq \Lambda_c$ with fundamental regions $\mathcal{V}, \mathcal{V}_c$ of volumes V, V_c (where $V \geq V_c$) respectively. Here Λ is called the *coarse* lattice which is a sublattice of Λ_c, the *fine* lattice. We denote the cardinality of a set A by $|A|$. The set $\mathcal{C}_{\Lambda_c,\mathcal{V}} = \{\Lambda_c \cap \mathcal{V}\}$ may be employed as the codebook for transmission over the AWGN channel, with coding rate R defined as $R = \frac{1}{n}\log|\mathcal{C}_{\Lambda_c,\mathcal{V}}| = \frac{1}{n}\log\frac{V}{V_c}$. Here $\rho = |\mathcal{C}_{\Lambda_c,\mathcal{V}}|^{\frac{1}{n}} = \left(\frac{V}{V_c}\right)^{\frac{1}{n}}$ is the nesting ratio of this *nested* (Λ, Λ_c) *lattice code pair*. A pair of *good* nested lattice codes, where Λ is both *Rogers good* and *Poltyrev good* and Λ_c is *Poltyrev good*, were shown to exist and be capacity achieving (as $n \to \infty$) for the AWGN channel [3]. Lattice code pairs may be extended to a nested lattice chain, which consists of nested lattice codes $\Lambda \subseteq \Lambda_1 \subseteq \Lambda_2$ which may be *Rogers* and *Poltyrev good* for arbitrary nesting ratios [15].

III. Lattice codes for the $(X+Z_1, X+Z_2)$ Wyner-Ziv model

Problem statement. We consider the lossy compression of the Gaussian source $Y = X + Z_1$, with side-information $X + Z_2$ available at the reconstruction node, where X, Z_1 and Z_2 are independent zero mean Gaussian random variables of variance P, N_1, and N_2 respectively. We note that, with slight abuse of notation, X, Z_1 and Z_2 denote n-dimensional vectors where n is the blocklength, or number of channel uses. The rate-distortion function for the source $X+Z_1$ taking on values in $\mathcal{X}_1$ with side-information $X+Z_2$ taking on values in $\mathcal{X}_2$ is defined as the minimum rate required to achieve a distortion D when $X+Z_2$ is available at the decoder. To be more specific, it is the infimum of rates R such that there exist maps $i_n : \mathcal{X}_1 \to \{1, 2, \cdots, 2^{nR}\}$ and $g_n : \mathcal{X}_2 \times \{1, 2, \cdots, 2^{nR}\} \to \mathcal{X}_1$ such that $\limsup_{n\to\infty} E[d(X+Z_1, g_n(X+Z_2, i_n(X+Z_1)))] \leq D$ for some distortion measure $d(\cdot,\cdot)$. If the distortion measure $d(\cdot,\cdot)$ is the squared error distortion, $d(X,\widehat{X}) = \frac{1}{n}E[||X-\widehat{X}||^2]$, then, by [16], the rate distortion function $R(D)$ for the source $X+Z_1$ given the side-information $X+Z_2$ is given by

$$R(D) = \frac{1}{2}\log\left(\frac{\sigma^2_{X+Z_1|X+Z_2}}{D}\right), \qquad 0 \leq D \leq \sigma^2_{X+Z_1|X+Z_2}$$
$$= \frac{1}{2}\log\left(\frac{N_1 + \frac{PN_2}{P+N_2}}{D}\right), \qquad 0 \leq D \leq N_1 + \frac{PN_2}{P+N_2},$$

and 0 otherwise, where $\sigma^2_{X+Z_1|X+Z_2}$ is the conditional variance of $X+Z_1$ given $X+Z_2$. We note that the lattice-code implementation of the Wyner-Ziv scheme of [5] considered the lossy compression of the source $X+Z$ with side-information X at the reconstruction node. In footnote 6 on pg. 1260 of [5] it is stated that this model may WLOG be used to capture all general jointly Gaussian sources and side-informations (including the aforementioned source $X+Z_1$ with side-information $X+Z_2$). The scheme we present next is an example of this more general scheme, and is provided only for completeness in order to use it to derive a lattice Compress-and-Forward scheme in Section IV.

Theorem 1. *The following rate-distortion function for the lossy compression of the source $X+Z_1$ subject to the reconstruction side-information $X+Z_2$ and squared error distortion metric may be achieved using lattice codes:*

$$R(D) = \frac{1}{2}\log\left(\frac{N_1 + \frac{PN_2}{P+N_2}}{D}\right), \qquad 0 \leq D \leq N_1 + \frac{PN_2}{P+N_2},$$

and 0 otherwise.

Proof: **General lattice Wyner-Ziv.** Consider a pair of nested lattice codes $\Lambda \subseteq \Lambda_q$, where Λ is Poltyrev-good with second moment $N_1 + \frac{PN_2}{P+N_2}$, and Λ_q is Rogers-good with

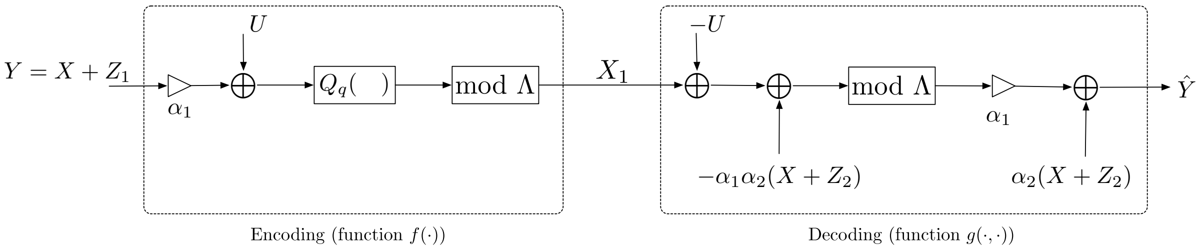

Fig. 1. Lattice coding for the $(X+Z_1, X+Z_2)$ Wyner-Ziv problem.

second moment D. We consider the encoding and decoding schemes of Fig. 1. We let U be a quantization dither signal which is uniformly distributed over $\mathcal{V}(\Lambda_q)$, and introduce the following MMSE coefficients (choices justified later):

$$\alpha_1 = \sqrt{1 - \frac{D}{N_1 + \frac{PN_2}{P+N_2}}}, \quad \alpha_2 = \frac{P}{P+N_2}. \tag{1}$$

Encoding. The encoder quantizes the scaled and dithered signal $\alpha_1(X+Z_1)+U$ to the nearest fine lattice point, which is then modulo-ed back to the coarse lattice Voronoi region as

$$\begin{aligned} I &= Q_q(\alpha_1(X+Z_1)+U) \mod \Lambda \\ &= (\alpha_1(X+Z_1)+U-E_q) \mod \Lambda, \end{aligned}$$

where $E_q := (\alpha_1(X+Z_1)+U) \mod \Lambda_q$ is independent of everything else and uniformly distributed over $\mathcal{V}(\Lambda_q)$ according to the Crypto lemma [13]. The encoder sends the index i of I to the decoder at the source coding rate

$$R = \frac{1}{n}\log\left(\frac{V(\Lambda)}{V(\Lambda_q)}\right) = \frac{1}{2}\log\left(\frac{N_1 + \frac{PN_2}{P+N_2}}{D}\right).$$

Decoding. The decoder receives the index i of I and reconstructs $\widehat{Y}$ as

$$\begin{aligned} \widehat{Y} &= \alpha_1((I - U - \alpha_1\alpha_2(X+Z_2)) \mod \Lambda) + \alpha_2(X+Z_2) \\ &= \alpha_1((\alpha_1((1-\alpha_2)X - \alpha_2 Z_2 + Z_1) - E_q) \mod \Lambda) \\ &\qquad + \alpha_2(X+Z_2) \\ &\equiv \alpha_1(\alpha_1((1-\alpha_2)X - \alpha_2 Z_2 + Z_1) - E_q) + \alpha_2(X+Z_2) \\ &= (\alpha_1^2 - \alpha_1^2\alpha_2 + \alpha_2)X + \alpha_2(1-\alpha_1^2)Z_2 + \alpha_1^2 Z_1 - \alpha_1 E_q, \end{aligned}$$

where the third equivalence is meant to denote asymptotic equivalence (as $n \to \infty$), since, as in [5]

$$\begin{aligned} Pr\{(\alpha_1((1-\alpha_2)X - \alpha_2 Z_2 + Z_1) - E_q) \mod \Lambda \\ \neq \alpha_1((1-\alpha_2)X - \alpha_2 Z_2 + Z_1) - E_q\} \to 0 \end{aligned} \tag{2}$$

goes to 0 as $n \to \infty$ for a sequence of a good nested lattice codes since

$$\begin{aligned} &\frac{1}{n}E||\alpha_1((1-\alpha_2)X - \alpha_2 Z_2 + Z_1) - E_q||^2 \\ &= \alpha_1^2\left(\frac{PN_2}{P+N_2} + N_1\right) + D = \frac{PN_2}{P+N_2} + N_1 = \sigma^2(\Lambda). \end{aligned} \tag{3}$$

The careful choice of the MMSE coefficients α_1 and α_2 as in (1) guarantees the above equation (3). Thus,

$$\begin{aligned} \widehat{Y} - Y &= (\alpha_1^2 - \alpha_1^2\alpha_2 + \alpha_2)X \\ &\quad + \alpha_2(1-\alpha_1^2)Z_2 + \alpha_1^2 Z_1 - \alpha_1 E_q - (X+Z_1) \\ &= -(1-\alpha_1^2)(1-\alpha_2)X + \alpha_2(1-\alpha_1^2)Z_2 - (1-\alpha_1^2)Z_1 - \alpha_1 E_q \\ &= -(1-\alpha_1^2)((1-\alpha_2)X - \alpha_2 Z_2 + Z_1) - \alpha_1 E_q, \end{aligned}$$

from which we may bound the squared error distortion as

$$\frac{1}{n}E||\widehat{Y} - Y||^2 = (1-\alpha_1^2)^2\left(\frac{PN_2}{P+N_2} + N_1\right) + \alpha_1^2 D = D,$$

again through the careful choice of α_1 and α_2 as in (1). ■

Remarks on the MMSE coefficients α_1 and α_2. We first note that the source $X+Z_1$ may be expressed as

$$X + Z_1 = \alpha_2(X+Z_2) + (1-\alpha_2)X + Z_1 - \alpha_2 Z_2,$$

and that by choosing $\alpha_2 = \frac{P}{P+N_2}$, $X+Z_2$ and $(1-\alpha_2)X + Z_1 - \alpha_2 Z_2$ may be shown to be independent. In this case, we are able to equate α_2 with the a of footnote 6 on pg. 1260 of [5], thereby relating the above scheme to that of [5]. In this case, we may intuitively think of α_1 as a source coding MMSE coefficient, and of α_2 as a channel coding MMSE coefficient, since it plays a role similar to the MMSE coefficient used in the lattice channel coding problem [3], i.e. it minimizes $E[(1-\alpha_2)X - \alpha_2 Z_2]^2$. In particular, we may see the importance of the correct choice of these coefficients by considering the alternative choices of α_1 and α_2, with the corresponding suboptimal rates:

- If α_1 is set to 1, the second moment of the coarse lattice is changed accordingly, the rate distortion function is

$$R(D) = \frac{1}{2}\log\left(1 + \frac{N_1 + \frac{PN_2}{P+N_2}}{D}\right) > \frac{1}{2}\log\left(\frac{N_1 + \frac{PN_2}{P+N_2}}{D}\right).$$

- If α_2 is set to 1, the second moment of the coarse lattice is changed accordingly, the rate distortion function is

$$R(D) = \frac{1}{2}\log\left(\frac{N_1 + N_2}{D}\right) > \frac{1}{2}\log\left(\frac{N_1 + \frac{PN_2}{P+N_2}}{D}\right).$$

IV. Lattice coding for Compress-and-Forward

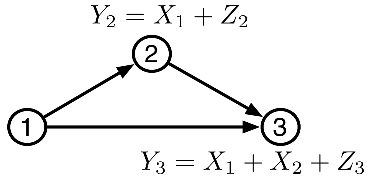

Fig. 2. Three node Gaussian relay channel

Using the general lattice-coding based Wyner-Ziv problem of the previous Section, we now implement a lattice Compress-and-Forward scheme for the three node Gaussian relay channel. The model is shown in Fig.2, where the transmitter (Node 1) and the relay (Node 2) may transmit $X_1 \in \mathcal{X}_1$ and $X_2 \in \mathcal{X}_2$ subject to power constraints $E[|X_1|^2] \leq P_1$, $E[|X_2|^2] \leq P_2$, and $Y_2 \in \mathcal{Y}_2$ and $Y_3 \in \mathcal{Y}_3$ are the output random variables which are related to the inputs through the relationships in Fig. 2, where Z_2, Z_3 are independent additive white Gaussian noise of variance N_2, N_3. Furthermore, let $X_i(j)$ denote node i's input at the j-th channel use, and let $X_1^n := (X_1(1), X_1(2), \cdots, X_1(n))$. Similar notation is used for received signals $Y_i(j)$. In this channel coding problem, we use classic definitions for achievable rates, i.e. a $(2^{nR}, n)$ code for a relay channel consists of a set of integers $\mathcal{W} = \{1, 2, \cdots, 2^{nR}\}$, an encoding function $X_1 : \{1, 2, \cdots, 2^{nR}\} \to \mathcal{X}_1^n$, a set of relay functions $\{f_i\}_{i=1}^n$such that $x_{2i} = f_i(Y_2(1), Y_2(2), \cdots, Y_2(i-1)), \quad 1 \leq i \leq n$, and a decoding function $g : \mathcal{Y}_3^n \to \{1, 2, \cdots 2^{nR}\}$. We let the probability of error of this $(2^{nR}, n)$ code be defined as $P_e^{(n)} := \frac{1}{2^{nR}} \sum_{w \in \mathcal{W}} Pr\{g(Y_3^n) \neq w | w \text{ sent}\}$, for $w \in \mathcal{W}$. The rate R is said to be achievable if there exists a sequence of $(2^{nR}, n)$ codes such that $P_e^{(n)} \to 0$ as $n \to \infty$.

The Compress-and-Forward (CF) scheme originally proposed in [2] utilizes a random coding argument, block Markov encoding, Wyner-Ziv binning, and simultaneous joint typicality decoding. Our goal is to replace random codes with lattice codes and change the achievability techniques accordingly.

Theorem 2. *For the three user Gaussian relay channel described by the input/output equations $Y_2 = X_1 + Z_2$ and $Y_3 = X_1 + X_2 + Z_3$, with corresponding input and noise powers P_1, P_2, N_2, N_3, the following rate may be achieved using lattice codes in a lattice Compress-and-Forward fashion:*

$$R < \frac{1}{2} \log \left(1 + \frac{P_1}{N_3} + \frac{P_1 P_2}{P_1 N_2 + P_1 N_3 + P_2 N_2 + N_2 N_3} \right).$$

Proof: **Lattice codebook construction.** We employ three "good" lattice codebooks, where we drop all subscripts / superscripts n for ease of exposition and note that all lattices and lattice points are n-dimensional.

• Channel codebook for Node 1: codewords t_1 in codebook $\mathbf{C_1} = \{\Lambda_{c1} \cap \mathcal{V}(\Lambda_1)\}$ where $\Lambda_1 \subseteq \Lambda_{c1}$ is a pair of good nested lattice codes – Λ_1 is both Rogers-good and Poltyrev-good and Λ_{c1} is Poltyrev-good. We set $\sigma^2(\Lambda_1) = P_1$ to satisfy the transmitter power constraint. We associate each message $w \in \mathcal{W}$ with the codeword t_1 in one-to-one fashion, $w \leftrightarrow t_1$, and send a dithered version of t_1. Note that $|\mathbf{C_1}| = 2^{nR}$.

• Channel codebook for Node 2: codewords t_2 in codebook $\mathbf{C_2} = \{\Lambda_{c2} \cap \mathcal{V}(\Lambda_2)\}$ where $\Lambda_2 \subseteq \Lambda_{c2}$ is a pair of good nested lattice codes: – Λ_2 is both Rogers-good and Poltyrev-good and Λ_{c2} is Poltyrev-good. We set $\sigma^2(\Lambda_2) = P_2$ to satisfy the relay power constraint. We associate each compression index i with the codeword t_2 in one-to-one fashion: $i \leftrightarrow t_2$, and send a dithered version of t_2. Note that $|\mathbf{C_2}| = 2^{nR'}$.

• Quantization/Compression codebook: $t_q \in \mathbf{C_q} = \{\Lambda_q \cap \mathcal{V}(\Lambda)\}$ where $\Lambda \subseteq \Lambda_q$ is a pair of good nested lattice codes – Λ is Poltyrev-good and Λ_q is Rogers-good. We set $\sigma^2(\Lambda_q) = D$, $\sigma^2(\Lambda) = N_2 + \frac{P_1 N_3}{P_1 + N_3} + D$, such that the source coding rate is $\widehat{R} = \frac{1}{2} \log \left(1 + \frac{N_2 + \frac{P_1 N_3}{P_1 + N_3}}{D} \right)$.

Encoding. We use block Markov encoding as in [2]. In block j, Node 1 chooses the codeword $t_1(j)$ associated with the message $w(j)$ to be transmitted in block j and transmits

$$X_1(j) = (t_1(j) + U_1(j)) \mod \Lambda_1,$$

where $U_1(j)$ is the dither uniformly distributed over $\mathcal{V}(\Lambda_1)$. Node 2 quantizes the received signal in the last block $j-1$, $Y_2(j-1) = X_1(j-1) + Z_2(j-1)$ to $I(j-1)$ (with index $i(j-1)$) by using the quantization lattice code pair (Λ_q, Λ) as described in the encoding part of Section III, where we set $\alpha_1 = 1$ and we set the second moment of Λ to be $\sigma^2(\Lambda) = N_2 + \frac{P_1 N_3}{P_1 + N_3} + D$. These settings will be explained later. Node 2 chooses the codeword $t_2(j-1)$ associated with the index $i(j-1)$ of $I(i-1)$ and sends

$$X_2(j) = (t_2(j-1) + U_2(j)) \mod \Lambda$$

with U_2 the dither signal uniformly distributed over $\mathcal{V}(\Lambda_2)$.

Decoding. In block j, Node 3 receives

$$Y_3(j) = X_1(j) + X_2(j) + Z_3(j).$$

It first decodes $t_2(j-1)$, and then the associated $I(j-1)$ and $X_2(j)$, using lattice decoding as in [3] subject to the channel coding rate constraint (recall that t_2 is of rate R')

$$R' < \frac{1}{2} \log \left(1 + \frac{P_2}{P_1 + N_3} \right),$$

which ensures the correct decoding of $t_2(j-1)$. We note that the source coding rate of I, $\widehat{R}$ must be less than the channel coding rate R', which means

$$\frac{1}{2} \log \left(1 + \frac{N_2 + \frac{P_1 N_3}{P_1 + N_3}}{D} \right) < \frac{1}{2} \log \left(1 + \frac{P_2}{P_1 + N_3} \right). \quad (4)$$

Node 3 sutracts the decoded $X_2(j)$ from $Y_3(j)$ and obtains

$$Y_3'(j) = Y_3(j) - X_2(j) = X_1(j) + Z_3(j)$$

which is used as direct-link side-information in the next block $j+1$. In the previous block, Node 3 had also obtained $Y_3'(j-1) = X_1(j-1) + Z_3(j-1)$. Combining this with $I(j-1)$, Node 3 uses $Y_3'(j-1)$ as side-information to reconstruct $\widehat{Y}_2(j-1)$ as in the decoding part of Section III, with $\alpha_1 = 1$, and $\sigma^2(\Lambda) = N_2 + \frac{P_1 N_3}{P_1 + N_3} + D$.

Thus, we see that the CF scheme employs the $(X + Z_1, X + Z_2)$ Wyner-Ziv coding scheme of Section III where the source to be compressed at the relay is $X_1 + Z_2$ and the side-information at the receiver (from the previous block) is $X_1 + Z_3$. One small difference from what was described in Section III is that X_1 is not strictly Gaussian distributed for finite n. However, X_1 will approach a Gaussian random variable as $n \to \infty$ since Λ_1 is Rogers-good. The step in (2) in Section III now corresponds to

$$P_{e,n} = Pr\{(((1-\alpha_2)X_1 - \alpha_2 Z_3 + Z_2) - E_q) \mod \Lambda \neq ((1-\alpha_2)X_1 - \alpha_2 Z_3 + Z_2) - E_q\}$$

since we have chosen $\alpha_1 = 1$, and since

$$\frac{1}{n}E||(1-\alpha_2)X_1 - \alpha_2 Z_3 + Z_2 - E_q||^2 = \frac{P_1 N_3}{P_1 + N_3} + N_2 + D = \sigma^2(\Lambda). \quad (5)$$

Thus, the above error probability still goes to 0 as $n \to \infty$ since X_1, while not Gaussian in this case, may be treated as such as $n \to \infty$ as Λ_1 is Rogers-good. Essentially, X_1 may be treated just as E_q is treated. We also note that α_2 is chosen so as to guarantee (5).

The compressed $Y_2(j-1)$ may now be expressed as

$$\begin{aligned}\widehat{Y}_2(j-1) &= (\alpha_1^2 - \alpha_1^2\alpha_2 + \alpha_2)X_1(j-1) \\ &\quad + \alpha_2(1-\alpha_1^2)Z_3(j-1) + \alpha_1^2 Z_2(j-1) - \alpha_1 E_q(j-1) \\ &= X_1(j-1) + Z_2(j-1) - E_q(j-1)\end{aligned}$$

where $E_q = (Y_2 + U) \mod \Lambda$ (with U the quantization dither which is uniformly distributed over Λ) is independent and uniformly distributed over $\mathcal{V}(\Lambda_q)$ with second moment D. Now, Node 3 may decode $t_1(j-1)$ (and the associated $w(j-1)$) from $Y_3'(j-1)$ and $\widehat{Y}_2(j-1)$ by first linearly and coherently combining them as

$$\begin{aligned}&\frac{\sqrt{P_1}}{N_3}Y_3'(j-1) + \frac{\sqrt{P_1}}{N_2+D}\widehat{Y}_2(j-1) \\ &= \left(\frac{\sqrt{P_1}}{N_3} + \frac{\sqrt{P_1}}{N_2+D}\right)X_1(j-1) + \frac{\sqrt{P_1}}{N_3}Z_3(j-1) \\ &\quad + \frac{\sqrt{P_1}}{N_2+D}\left(Z_2(j-1) - E_q(j-1)\right).\end{aligned}$$

Since E_q will approach a Gaussian random vector of variance D as $n \to \infty$, the above equation may be treated as an AWGN channel. Using modulo lattice decoding [3], we may decode $t_1(j-1)$ (and the associated message $w(j-1)$) as long as

$$R < \frac{1}{2}\log\left(1 + \frac{P_1}{N_3} + \frac{P_1}{N_2+D}\right).$$

Combining this with the constraint (4), we obtain

$$R < \frac{1}{2}\log\left(1 + \frac{P_1}{N_3} + \frac{P_1 P_2}{P_1 N_2 + P_1 N_3 + P_2 N_2 + N_2 N_3}\right),$$

which is the CF rate achieved by Gaussian random codes in [9, pg.17–48]. ■

Remarks: Notice that there is a slight difference between the $(X + Z_1, X + Z_2)$ Wyner-Ziv coding scheme described in Section III and its application to the Compress-and-Forward scheme for the three node Gaussian relay channel. The reason we choose $\alpha_1 = 1$ rather than optimal coefficient $\alpha_1 = \sqrt{1 - \frac{D}{N_2 + \frac{P_1 N_3}{P_1 + N_3}}}$, and $\sigma^2(\Lambda) = N_2 + \frac{P_1 N_3}{P_1 + N_3} + D$ rather than $N_2 + \frac{P_1 N_3}{P_1 + N_3}$ is because we would like the quantization/compression error $\widehat{Y}_2 - Y_2$ to be independent of all other terms, so that we may view $\widehat{Y}_2 = X_1 + N_2 - E_q$ as an equivalent AWGN channel. This convention is generally used in Gaussian compress-and-forward such as in [9].

V. Conclusion

We have demonstrated a lattice Compress-and-Forward scheme for the three user Gaussian relay channel which achieves the same rate as that achieved using random Gaussian codebooks in Cover and El Gamal's CF rate for the Gaussian relay channel. Given the structured nature of lattice codes, this provides an alternative understanding of CF in Gaussian networks. This lattice CF scheme may pave the path to a more generic lattice-based achievability scheme for arbitrary networks, such as for example a structured version of the recent, general, noisy network coding scheme, or its combination with the recently introduced Compute-and-Forward framework. This is the subject of ongoing work.

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