Error Exponents of Parallel Two-way Discrete Memoryless Channels using Variable Length Coding

Kenneth Palacio-Baus*†, Meysam Asadi*, Natasha Devroye*
*University of Illinois at Chicago, {kpalac2, masadi, devroye}@uic.edu
†University of Cuenca, Ecuador

***Abstract*—Achievable error exponents for two-way parallel discrete memoryless channels (DMC) using variable block length coding (VLC) are presented. First, Forney's erasure decoding error exponent is shown to be achievable for both directions simultaneously. Next, for some rate-pairs, it is shown that the error exponent of the direction with a smaller capacity may be further increased by allocating feedback resources to it in the other direction, at the price of a decreased error exponent for the other terminal. The presented two-way communication scheme builds upon Draper-Sahai's one-way DMC achievability scheme with noisy feedback under VLC. Both achievable error exponent regions demonstrate that the use of VLC and interaction between the terminals may benefit both directions' error exponents over fixed block length and feedback free transmission.**[1]

I. Introduction

In one-way communications with feedback, variable length coding (VLC) techniques where decoding times are not determined before transmission have been shown to attain better error exponents than those achieved by fixed block length schemes, where decoding is done at pre-described times. The error exponent, or reliability, of a discrete memoryless channel (DMC) characterized by channel transition probabilities $p(y|x)$ under VLC for an average transmission rate $\bar{R}$ is defined as:

$$E(\bar{R}) = \lim_{\mathsf{E}[\Delta]\to\infty} -\frac{1}{\mathsf{E}[\Delta]} \log \mathsf{P}_e\left(\bar{R}, \Delta\right) \tag{1}$$

where $\mathsf{E}[\cdot]$ denotes expectation, Δ the transmission time (a random variable), and $\mathsf{P}_e\left(\bar{R}, \Delta\right)$ the probability of error at average rate $\bar{R}$ when decoding at time Δ.

Burnashev [1] showed that the error exponent using VLC coding over a DMC of capacity C with *complete output feedback* for an average transmission rate $\bar{R}$, denoted by $E_{\text{vl}}(\bar{R})$ is upper bounded by:

$$E_{\text{vl}}(\bar{R}) \leq E_{\text{Burn}}(\bar{R}) := C_1\left(1 - \bar{R}/C\right) \tag{2}$$

where, $C_1 = \max_{x_i,x_j}\left\{\sum_y p(y \mid x_i)\log\frac{p(y|x_i)}{p(y|x_j)}\right\}$ corresponds to the Kullback-Leibler divergence between the two most distinguishable symbols. A scheme able to achieve this bound was presented by Yamamoto and Itoh [2]. The reliability of DMCs can still be improved over the non-feedback reliability even if complete output feedback is not available, but at least a *single bit of noiseless feedback* is, and erasure decoding is employed. Forney [3] used this bit to tell the source whether decoding was successful (source sends a new message) or resulted in an erasure (source re-transmits the same message). Forney's reliability (3) is larger than the sphere packing upper bound ($E_{\text{sp}}(R)$ [4, Section 5.8]) for fixed block length coding without feedback for rate $\bar{R}$, given by

$$E_{\text{Forn}}(\bar{R}) = E_{\text{sp}}(\bar{R}) + C - \bar{R}. \tag{3}$$

In VLC schemes in general, potential errors may be corrected by retransmissions. Such retransmissions lead to higher error exponents, at the expense of variable decoding times and delays, which bring up the issue of synchronization – the transmitter and receiver must agree upon which message is currently being transmitted. If noiseless feedback is available (as in Burnashev, Yamatoto-Itoh, or Forney's models), then both terminals may easily stay synchronized. This is not the case if feedback is noisy: [5], [6] both address this and show that even in the presence of *noisy feedback*, reliability improvements over fixed block length coding are attainable in the one-way setting.

Here, we consider the non-previously studied *two-way* scenario where the destination is also interested in transmitting its own messages to the source. The transmitter and receiver may be referred to as Terminals 1 and 2. Error exponents for two-way channels were first studied in [7], using fixed block length coding over AWGN channels at zero-rate. Here we consider positive rates, and a *parallel two-way DMC,* where the channel transition probability breaks into the product of two one-way channels. This is a class of two-way channels relevant in time or frequency division systems, and is one of few classes of channels for which the two-way capacity region is known. We consider VLC and note that feedback is automatically noisy and shares the same channel as the data. This model thus pinpoints the tradeoff between allocating resources to feedback to help the other direction's error exponent versus using them to transmit one's own message. We show that interaction – using feedback – does improve the error exponent regions achievable, in contrast to the two-way parallel capacity region which is not improved by interaction [8].

II. Problem statement and contributions

A two-way DMC $(p(y_1y_2|x_1x_2), \mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}_1, \mathcal{Y}_2)$ is characterized by a set of channel transition probability mass functions $p(y_1y_2|x_1x_2)$, and finite sets of input and output alphabets $\mathcal{X}_i$ and $\mathcal{Y}_i$ for the i-th terminal, where $i = \{1, 2\}$ [8].

[1] This work was partially supported by NSF under awards 1053933 and 1645381. The contents of this article are solely the responsibility of the authors and do not necessarily represent the official views of the NSF.

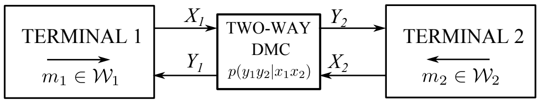

Fig. 1. Two-way discrete memoryless channel.

In the two-way *parallel* DMC $p(y_1y_2|x_1x_2) = p_{12}(y_2|x_1) \cdot p_{21}(y_1|x_2)$. Thus, each communication direction is characterized by $p_{i(3-i)}(y_{3-i}|x_i)$, and the two directions may be viewed as independent links operating in parallel. Terminals i and $3-i$ exchange, respectively, messages M_i, which are chosen at random uniformly from the sets $\mathcal{W}_i = \{1, 2, ...|\mathcal{W}_i|\}$.

Definition 1: A two-way variable length code used to exchange messages $M_1 \in \mathcal{W}_1$ and $M_2 \in \mathcal{W}_2$ over a two-way parallel DMC $(p_{12}(y_2|x_1) \cdot p_{21}(y_1|x_2), \mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}_1, \mathcal{Y}_2)$ comprises two sets of encoding functions:

$$x_{i,n} = \mathcal{W}_i \times \mathcal{Y}_i^{n-1} \to \mathcal{X}_i, \tag{4}$$

and two sets of decoding functions:

$$\phi_{i,n} : \mathcal{Y}_i^n \to \mathcal{W}_{3-i} \cup \{0\}, \tag{5}$$

for $n = 1, 2, \cdots$, and where 0 corresponds to an erasure. Eq. (4) shows how channel inputs at time n may depend on previously received channel outputs. We will refer to this as interaction / adaptation.

Random variable Δ corresponds to the time required to transmit and successfully decode both messages. Following [5], Δ is the first n for which messages in both directions are successfully decoded and not declared as an erasure

$$\Delta = n, \text{s.t.} \begin{cases} (\phi_{1_n}(y_1^n) \cdot \phi_{2_n}(y_2^n)) > 0, \\ (\phi_{1_n}(y_1^{n'}) \cdot \phi_{2_n}(y_2^{n'})) = 0, \ \forall n' < n \end{cases}. \tag{6}$$

Denote the average communication rate for each direction as $\bar{R}_{i(3-i)}$. Then, let $\left(\mathsf{P}_{e_{12}}\left(\left(\bar{R}_{12}, \bar{R}_{21}\right), \Delta\right), \mathsf{P}_{e_{21}}\left(\left(\bar{R}_{12}, \bar{R}_{21}\right), \Delta\right)\right)$ be the probability of error pair simultaneously attained for messages M_1, M_2, where:

$$\begin{aligned} &\mathsf{P}_{e_{i(3-i)}}\left(\left(\bar{R}_{12}, \bar{R}_{21}\right), \Delta\right) = \\ &\max_{m \in \mathcal{W}_i} \mathsf{P}\left(\phi_{3-i}\left(Y_{3-i}^{\Delta}\right) \neq m \mid M_i = m \text{ was sent }\right). \end{aligned} \tag{7}$$

A rate pair $\left(\bar{R}_{12}, \bar{R}_{21}\right)$ is called achievable if there exists a sequence of two-way variable length codes for which $\mathsf{P}_{e_{i(3-i)}}\left(\left(\bar{R}_{12}, \bar{R}_{21}\right), \Delta\right) \to 0$ and

$$\bar{R}_{i(3-i)} \leq \lim_{\mathsf{E}[\Delta] \to \infty} \frac{\log |\mathcal{W}_i|}{\mathsf{E}[\Delta]}. \tag{8}$$

The capacity region $\mathcal{C}$ of the two-way parallel DMC is the convex hull of the set of achievable rate pairs $(\bar{R}_{12}, \bar{R}_{21})$. For the parallel two-way channel, this corresponds to a rectangular region determined by the capacity of each link [8] – interaction does not increase the capacity region over the two one-way capacities. As shown in Fig. 2, the capacity of the $1 \to 2$ and $1 \leftarrow 2$ directions are denoted as C_{12} and C_{21}, respectively.

Our goal is to characterize the achievable error exponents for the probability of error pairs attained for a given average transmission rate pair and transmission time Δ as:

Definition 2: A pair of error exponents $\left(E_{\mathrm{vl}_{12}}\left(\bar{R}_{12}, \bar{R}_{21}\right), E_{\mathrm{vl}_{21}}\left(\bar{R}_{12}, \bar{R}_{21}\right)\right)$ is called achievable for a corresponding average rate pair $\left(\bar{R}_{12}, \bar{R}_{21}\right)$ over a two-way parallel DMC, if there exists a two-way VLC code such that for large Δ, simultaneously:

$$-\frac{1}{\mathsf{E}[\Delta]} \log \mathsf{P}_{e_{12}}\left(\left(\bar{R}_{12}, \bar{R}_{21}\right), \Delta\right) \geq E_{\mathrm{vl}_{12}}\left(\bar{R}_{12}, \bar{R}_{21}\right) \tag{9}$$

$$-\frac{1}{\mathsf{E}[\Delta]} \log \mathsf{P}_{e_{21}}\left(\left(\bar{R}_{12}, \bar{R}_{21}\right), \Delta\right) \geq E_{\mathrm{vl}_{21}}\left(\bar{R}_{12}, \bar{R}_{21}\right). \tag{10}$$

We note that each achievable average rate-pair is associated with an error exponent region, formed by the union over all the achievable error exponent pairs. We will show that the capacity region can be divided into regimes where different error exponents are achievable.

A. Contributions

We present an initial characterization of error exponents of the two-way parallel DMC under VLC. First, we apply results for the one-way channel under fixed block length coding (FLC) for each direction without feedback, as comparison points for the benefits of VLC and interaction / adaptation. Let the random coding error exponent without feedback at rate R be denoted as $E_r(R)$ [4, Section 5.6]. Then,

Proposition 1: An achievable error exponent pair, for a communication rate pair $(\bar{R}_{12}, \bar{R}_{21})$, is:

$$E_{\mathrm{vl}_{12}}(\bar{R}_{12}, \bar{R}_{21}) \geq E_{\mathrm{r}}(\bar{R}_{12}), \tag{11}$$

$$E_{\mathrm{vl}_{21}}(\bar{R}_{12}, \bar{R}_{21}) \geq E_{\mathrm{r}}(\bar{R}_{21}), \tag{12}$$

for $C_{0_{12}} < \bar{R}_{12} < C_{12}$ and $C_{0_{21}} < \bar{R}_{21} < C_{21}$, where $C_{0_{i(3-i)}}$ is the zero-error capacity for each direction without feedback. At rates $\bar{R}_{12} < C_{0_{12}}$ and $\bar{R}_{21} < C_{0_{21}}$, zero error is achievable and the error exponents are infinite.

The error exponents attainable at each terminal under VLC may be upper bounded by Burnashev's exponent [1] by providing perfect output feedback to each terminal:

Proposition 2: The error exponent region of a two-way parallel DMC operating at an average communication rate pair $(\bar{R}_{12}, \bar{R}_{21})$ under VLC is bounded by:

$$E_{\mathrm{vl}_{12}}(\bar{R}_{12}, \bar{R}_{21}) \leq E_{\mathsf{Burn}}(\bar{R}_{12}) \tag{13}$$

$$E_{\mathrm{vl}_{21}}(\bar{R}_{12}, \bar{R}_{21}) \leq E_{\mathsf{Burn}}(\bar{R}_{21}). \tag{14}$$

The next proposition – our main contribution – is based on the techniques proposed by Draper and Sahai in [5] for the one-way DMC with noisy feedback. We will show how this scheme may be adapted to support two-way communications, a scenario for which the achievable error exponents have not yet been studied. Draper and Sahai's "Basic+Erasure" (BE) in [5] is most suited to the two-way setting, since the backward link can be tweaked to transmit messages in addition to feedback, as we describe in Section III. Moreover, this scheme can always achieve an error exponent that performs at least as well as Forney's (3), and that improves upon it when the feedback channel is stronger than the forward channel. We assume, as Draper and Sahai do, that the forward and backward links have fixed block length and variable length zero-error capacity with and without feedback equal to zero[2]. The reliability function of the scheme is presented in [5, Section 4.2, Eq. (19)] and depends upon a parameter γ, defined in [5, Eq. (15)], which establishes "high-rate" and "low-rate" regimes split by the critical rate $\bar{R}^*$ defined in [5, Eq. (16)].

In the two-way scenario, we characterize error exponents for two average rate-pair regimes. The small constant parameter $\delta > 0$ characterizes the length of a synchronization anytime code, as explained soon, see Fig. 3.

Rate pair regime (i): the complete capacity region

$$0 \leq \bar{R}_{12} \leq (1-\delta) C_{12} \tag{15}$$

$$0 \leq \bar{R}_{21} \leq (1-\delta) C_{21} \tag{16}$$

where terminals cooperate to achieve Forney's error exponent in each direction, as the rate of the anytime code used in Draper and Sahai's scheme that achieves the Forney error exponent for noisy feedback may be made arbitrarily small;

Rate-pair regime (ii): the shaded area in Fig. 2, where the (assumed to be weaker) $1 \to 2$ direction attains the BE scheme error exponent, while the (assumed to be stronger) $1 \leftarrow 2$ direction is able to send its own messages using Forney's technique with a decreased error exponent. This region is

$$\bar{R}_{21} \leq C_{21}\left(1 - \delta - \frac{\bar{R}_{12}}{R^*_{\text{data}_{12}}} + (1-\gamma)\frac{\bar{R}_{12}}{R^*_{\text{data}_{12}}}\right), \quad \text{for } 0 \leq \bar{R}^*_{12} \leq (1-\delta) C_{12}, \tag{17}$$

where γ is adapted from [5, Eq. (15)] and is given by

$$\gamma = \min\left\{1, \left((1-\delta)\frac{R^*_{\text{data}_{12}}}{\bar{R}_{12}} - 1\right)\frac{C_1}{C_{21}}\right\}, \tag{18}$$

$\bar{R}^*_{12}$ is adapted from [5, Eq. (16)] and given by $\bar{R}^*_{12} = \frac{C_{12}(1-\delta)}{1+\frac{C_{21}}{C_1}}$ and $R^*_{\text{data}_{12}}$, as we will show later, is optimized for each operation regime of the BE scheme (low and high rate in the $1 \to 2$ direction) and given by

$$R^*_{\text{data}_{12}} = \begin{cases} \text{Eq. (25)}, & 0 \leq \bar{R}_{12} \leq \bar{R}^*_{12} \\ C_{12}, & \bar{R}^*_{12} < \bar{R}_{12} \leq (1-\delta) C_{12} \end{cases}. \tag{19}$$

This leads to our main contribution:

Proposition 3: An achievable error exponent pair for the two-way parallel DMC under VLC for an average rate-pair $(\bar{R}_{12}, \bar{R}_{21})$ is:

Under rate-pair regime (i):

$$E_{\text{vl}_{12}} \geq E_{\text{Forn}}\left(\frac{\bar{R}_{12}}{1-\delta}\right) \tag{20}$$

$$E_{\text{vl}_{21}} \geq E_{\text{Forn}}\left(\frac{\bar{R}_{21}}{1-\delta}\right) \tag{21}$$

[2]The VLC zero-error capacity region of the two-way parallel is open to the best of our knowledge. For rates inside this region the error exponents are infinite. From [9, Theorem 3] note that the variable-length zero-error capacity of a forward DMC with noisy feedback, C_0^{VL-NF}, is lower bounded by the zero-undetected-error capacity [3] of the forward channel if both the forward and backward channels have positive zero-undetected-error capacities.

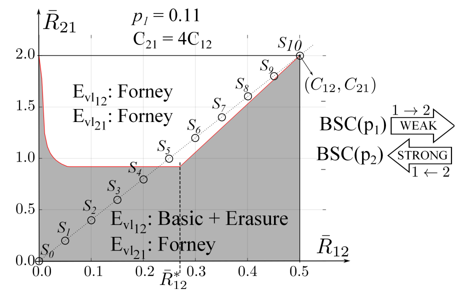

Fig. 2. Two-way parallel DMC error exponent regimes, for $C_{21} = 4C_{12}$. Points S_i for $i = 0, ..., 10$, denote rate pairs $(\bar{R}_{12}, \bar{R}_{21})$ along the diagonal of the capacity region which are used in the numerical simulation of Fig. 4.

Under rate-pair regime (ii):

$$E_{\text{vl}_{12}} \geq \frac{E_{\text{forn}}\left(R^*_{\text{data}_{12}}\right) + \min\left\{\left(1-\delta-\frac{\bar{R}_{12}}{R^*_{\text{data}_{12}}}\right)C_1, \gamma\frac{\bar{R}_{12}}{R^*_{\text{data}_{12}}}C_{21}\right\}}{1+\gamma\frac{\bar{R}_{12}}{R^*_{\text{data}_{12}}} + \left(1-\delta-\frac{\bar{R}_{12}}{R^*_{\text{data}_{12}}}\right)} \tag{22}$$

$$E_{\text{vl}_{21}} \geq \frac{E_{\text{forn}}\left(\frac{\bar{R}_{21}}{1-\delta-\frac{\bar{R}_{12}}{R^*_{\text{data}_{12}}}+(1-\gamma)\frac{\bar{R}_{12}}{R^*_{\text{data}_{12}}}}\right)}{1+\gamma\frac{\bar{R}_{12}}{R^*_{\text{data}_{12}}} + \left(1-\delta-\frac{\bar{R}_{12}}{R^*_{\text{data}_{12}}}\right)}. \tag{23}$$

The error exponent in the weaker ($1 \to 2$) direction corresponds to that attained by the BE scheme for the one-way channel in [5] and is thus greater than the sphere packing upper bound determined by fixed block length coding. This improvement comes at the price of a reduction in the error exponent of the stronger direction ($1 \leftarrow 2$). This decrease originates from 1) a large delay imposed by the other direction, see Eq. (6); and 2) a reduced number of channel uses destined for message transmission in the $1 \leftarrow 2$ direction (as feedback uses the rest). Thus, the instantaneous data rate $R_{\text{data}_{21}}$ is higher than the average rate $\bar{R}_{21}$ yielding a smaller value when (3) is evaluated.

III. Two-way achievability scheme construction

We now present the scheme that achieves Proposition 3. Our scheme extends Draper-Sahai's one-way, with noisy feedback scheme to support two-way communication. This involves resolving two-way synchronization issues and how forward and feedback resources may be effectively shared.

To address synchronization, we replicate their approach and use anytime codes in both directions. Each terminal has a finite number L distinct stacks of messages, which are sent from one at a time in round-robin scheme using slots of N channel uses. A round-robin scheduling of time slots of length N as in [5, Fig. 3] is depicted in Fig. 3. A new transmission or retransmission of the message may happen only after $(L-1)N$ channel uses. An anytime code is fed back to each terminal during each message transmission, carrying an L bit message of decoding decisions of the L stacks [5].

This code allows transmitters to receive an updated decisions vector of all L stacks every time a message is transmitted, and hence the reliability increases with L. Draper and

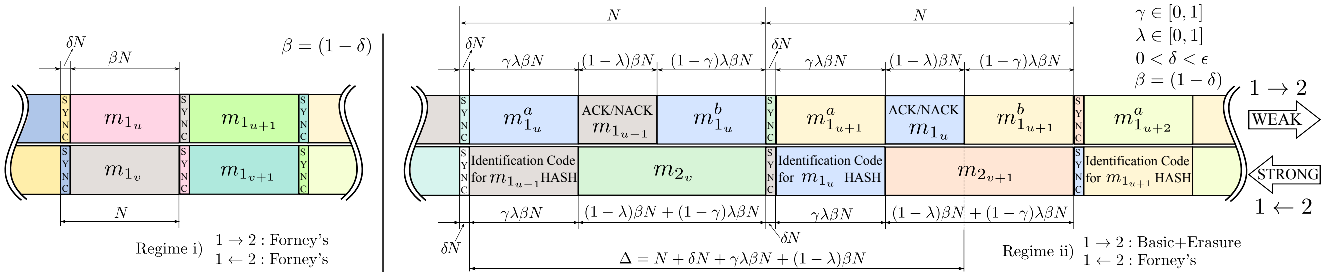

Fig. 3. Two-way DMC achievability scheme block diagram for average rate-pair regimes (i) and (ii).

Sahai demonstrated that for L large enough [5, Eq. (31)], and not a function of N, synchronization may be guaranteed. In our scheme, each bit communicated using this anytime code utilizes δN channel uses, thus, the rate of the anytime is zero as long as δ is chosen to satisfy $\lim_{N\to\infty} \frac{1}{\delta N} = 0$.

A. Error Exponents for Rate-pair Regime (i)

In this regime, Forney's error exponent (3) can be achieved in each direction, transmitting $2^{\beta N R_{\text{data}_{12}}}$ and $2^{\beta N R_{\text{data}_{21}}}$ messages respectively, by using the block diagram on the left side of Fig. 3. Note first that the transmission time $\Delta = (1-\delta)N$, thus $\lim\limits_{N\to\infty} \frac{\mathsf{E}[\Delta]}{N} \approx 1$. Each direction allocates an arbitrarily small fraction δ of the time slot length N to the anytime synchronization code. The remaining $(1-\delta)N$ channel uses are used for message transmission. This setting enables each direction to achieve (3) since both sources learn the decoding status of each message with increasing reliability that depends on the choice of L, see [5, Section 4.2]. Finally, Eq.s (20) and (21) in Proposition 3 result from direct application of (3), evaluated at the instantaneous transmission rates.

B. Error Exponents for Rate-pair Regime (ii)

In this regime, we use the block diagram on the right side of Fig. 3, which extends the BE scheme in [5] to support two-way communication. Here, the weaker $1 \to 2$ direction uses the BE scheme for $2^{\lambda\beta N R_{\text{data}_{12}}}$ messages , whereas the stronger $1 \leftarrow 2$ direction communicates using Forney's erasure decoding technique for $2^{[(1-\lambda)\beta N + (1-\gamma)\lambda\beta N] R_{\text{data}_{21}}}$ messages. Only the weaker direction is suitable for the BE scheme as the BE scheme requires a strong feedback channel (relative to the forward channel) to operate, which is only possible for one of the two directions. The scheme adaptation for two-way results by noting that in [5, Sections 4.1-2, Figs. 2-3] we can allocate up to $(1-\lambda)(1-\delta)N + (1-\gamma)\lambda(1-\delta)N$ channel uses to message transmission in the feedback direction ($1 \leftarrow 2$ in this work): the first $(1-\lambda)(1-\delta)N$ using the time the feedback channel remains *idle*, and the remaining $(1-\gamma)\lambda(1-\delta)N$, by shortening the synchronization anytime code sent over the feedback channel to an arbitrarily small fraction of length δN.

Since the BE scheme is directly applied for the $1 \to 2$ link (see Fig. 3), the asymptotically approachable expected delay is shown in Eq. (24), [5, Eq. (11) and Appendix].

$$\lim_{N\to\infty} \frac{\mathsf{E}[\Delta]}{N} = 1 + [\gamma\lambda + (1-\lambda)]\,(1-\delta)\,. \tag{24}$$

Also, note that a message from terminal 1, transmitted at a given time u in this direction (m_{1_u}) is divided into two chunks, denoted by $m^a_{1_u}$ and $m^b_{1_u}$, and that the hash message returned from terminal 2 refers to the complete message m_{1_u}.

1) Error exponent for the $1 \to 2$ direction: Assuming this direction is weaker that the other, its error exponent is determined by the BE scheme, [5, Section 4.2, Eq. (18)] which we have adapted to Eq. (22). There are $\lambda\beta N R_{\text{data}_{12}}$ bits of data transmitted per time slot. Since the probability of retransmission tends to zero as N becomes large, the average rate $\lim_{N\to\infty} \bar{R}_{12} = \lambda R_{\text{data}_{12}}(1-\delta)$, since $\beta = 1-\delta$.

As [5] indicates, the denominator of (22) is determined by (24), and the numerator comprises the contribution to the probability of error of erasure decoding $E_{\text{Forn}}(R^*_{\text{data}_{12}})$ ("Erasure" part of the scheme), and a term related to the contributions of a missed NACK, and one of a hash message collision ("Basic" part of the scheme). Then, the BE scheme is determined by finding the optimal value of $R^*_{\text{data}_{12}}$ such that the largest Error exponent is obtained in each regime:

(i) The "High-rate" regime: $\bar{R}^*_{12} \le \bar{R}_{12} \le (1-\delta)C_{12}$, corresponds to the use of the *Basic* scheme [5, Eq. (14)]; The largest error exponent is obtained setting $R^*_{\text{data}_{12}} = C_{12}$.

(ii) The "Low-rate" regime: $0 \le \bar{R}_{12} \le \bar{R}^*_{12}$, attains the largest error exponent for $R^*_{\text{data}_{12}}(\bar{R}_{12})$ determined by:

$$R^*_{\text{data}_{12}}(\bar{R}_{12}) = \underset{\bar{R}_{12} \le R_{\text{data}_{12}} \le (1-\delta)C_{12}}{\text{argmax}} \frac{E_{\text{forn}}(R_{\text{data}_{12}}) + \min\left\{(1-\delta-\frac{\bar{R}_{12}}{R_{\text{data}_{12}}})C_1, \gamma\frac{\bar{R}_{12}}{R_{\text{data}_{12}}}C_{21}\right\}}{1+\gamma\frac{\bar{R}_{12}}{R_{\text{data}_{12}}} + \left(1-\delta-\frac{\bar{R}_{12}}{R_{\text{data}_{12}}}\right)}. \tag{25}$$

Note from (18) and $\lambda = \frac{\bar{R}_{12}}{(1-\delta)R_{\text{data}_{12}}}$, that both γ and λ are optimized in (25) since they are functions of $R_{\text{data}_{12}}$. This optimization sets λ and γ to maximize the error exponent in the weaker direction, and dictates the number of channel uses remaining for message transmission in the opposite direction.

2) Error exponent for the $1 \leftarrow 2$ direction: is characterized by Forney's erasure decoding [3]; Eq. (23) is obtained from Δ, and E_{Forn} from (3). The latter is evaluated at an instantaneous rate that is higher than the average communication rate $\bar{R}_{21}$, as messages in the $1 \leftarrow 2$ direction are transmitted using block length $(1-\lambda)(1-\delta)N + (1-\gamma)\lambda(1-\delta)N < N$.

IV. Numerical Simulation

Fig. 4 presents a numerical evaluation of the error exponent regions simultaneously achieved in both directions for the set of rate-pairs $S_0 - S_{10}$ in the capacity region, as shown in Fig. 2. We evaluated a two-way parallel DMC consisting of two BSC(p_i) channels (each with a zero-error capacity of zero), with parameters $p_1 = 0.215$ and $p_2 = 0.000011$ for directions $1 \to 2$ and $1 \leftarrow 2$ respectively. Note C_{21} is about four times the capacity of the C_{12}. The rate axis in the three dimensional plot has been normalized for both directions. For Rate-pair Regime (i), Forney's error exponent can be achieved in both directions as illustrated in dark dotted blue. The solid magenta line shows the BE error exponent for the $E_{\text{vl}_{12}}$ as in [5].

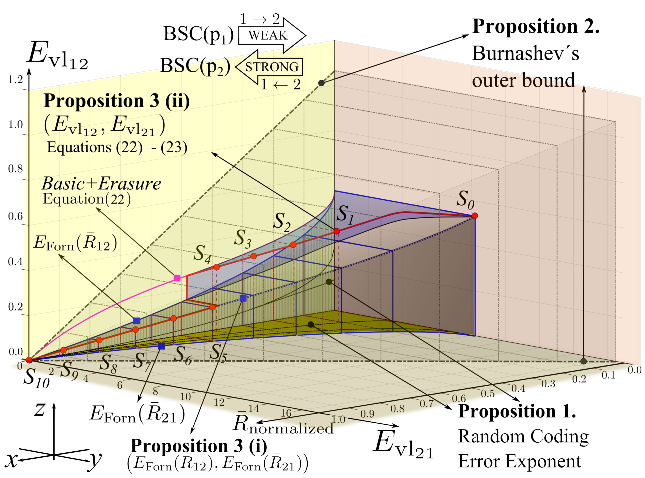

Fig. 4. Achievable error exponents in both directions. x-axis shows the normalized channel capacity, and the y and z axes the error exponents of the two directions. The black line above each line of Proposition 1 is the sphere packing upper bound for each direction that lies below Forney's bound. The red circles indicate rate-pairs S_i taken as shown in Fig. 2.

Note that for points S_0 to S_4, the $1 \to 2$ direction is able to achieve the BE error exponent, while the $1 \leftarrow 2$ direction's error exponent corresponds to Forney's error exponent evaluated at a higher instantaneous rate, therefore smaller than that when both directions operate in regime (i), as is the case for points S_5 to S_{10}. Thus, the $1 \to 2$ direction does not achieve the BE exponent, and both achieve Forney's error exponent.

The solid red line indicates all the simultaneously achieved error exponent pairs for rate pairs on the diagonal from $(0, 0)$ to (C_{12}, C_{21}). These may be compared with the inner bounds from Proposition 1 and the outer bounds from Proposition 2.

V. Conclusions

We have derived error exponent regions for the two-way parallel DMC, which demonstrate that the use of VLC benefits the error exponents of both directions, as they can simultaneously achieve Forney's error exponent, thus beating fixed block coding schemes at all rates. The error exponents achieved by rate pairs in Rate-pair Regime (ii) show that under VLC, the operating rate-pair determines whether interaction could benefit the error exponent of the weaker direction at the cost of a decreased error exponent for the stronger direction.

We note that Sato-Yamamoto's scheme under VLC for the one-way DMC with noisy feedback [6] exhibits a significantly better performance than the Draper-Sahai scheme under equivalent channels conditions. However, the Sato-Yamamoto feedback coding strategy leaves no explicit room in the feedback direction for message transmission in a two-way scheme. Whether it can be adapted to the two-way scenario successfully requires further research.

Appendix

Eq. (17) characterizes the average rate-pair regime (ii) shaded area in Fig. 2. The shape of this sub-region can be derived by noting that the error exponents for each direction are connected through the number of channel uses assigned to message transmission and feedback in the stronger direction. This allocation depends upon the choice of parameters λ and γ (see Fig. 3) and determined by $\bar{R}_{12}$ and R^*_{data}. Observe that $\bar{R}_{21}$ is constrained by $\bar{R}_{12}$ when the BE is used, since in the numerator of (23), the argument of $E_{\text{forn}}(R)$ cannot exceed the channel's capacity C_{21}:

$$\frac{\bar{R}_{21}}{1 - \delta - \frac{\bar{R}_{12}}{R^*_{\text{data}_{12}}} + (1-\gamma)\frac{\bar{R}_{12}}{R^*_{\text{data}_{12}}}} \leq C_{21}. \tag{26}$$

Solving this for $\bar{R}_{21}$ yields Eq. (17). This result implies that the BE scheme can operate at rate $\bar{R}_{12}$ in the $1 \to 2$ direction, only if the desired operation rate in the $1 \leftarrow 2$ direction $\bar{R}_{21}$ satisfies (26). In the alternate case, Forney's scheme must be used in both directions, and the system would operate in Rate-pair Regime (i). Fig. 2 was obtained by evaluating Eq. (17) for $0 \leq \bar{R}_{12} \leq (1-\delta)C_{12}$, by setting $R^*_{\text{data}_{12}} = C_{12}$ in the high-rate regime, and using Eq. (25) (numerically evaluated) for the low-rate regime.

References

[1] M. V. Burnashev, "Data transmission over a discrete channel with feedback: Random transmission time," *Problemy peredachi informatsii*, vol. 12, no. 4, pp. 10–30, 1976.

[2] H. Yamamoto and K. Itoh, "Asymptotic performance of a modified schalkwijk-barron scheme for channels with noiseless feedback (corresp.)," *IEEE Transactions on Information Theory*, vol. 25, no. 6, pp. 729–733, November 1979.

[3] G. Forney, "Exponential error bounds for erasure, list, and decision feedback schemes," *IEEE Transactions on Information Theory*, vol. 14, no. 2, pp. 206–220, Mar. 1968.

[4] R. G. Gallager, *Information Theory and Reliable Communication*. New York, NY: Wiley, 1968.

[5] S. C. Draper and A. Sahai, "Variable-length channel coding with noisy feedback," *European Transactions on Telecommunications*, vol. 19, no. 4, pp. 355–370, Jun. 2008.

[6] A. Sato and H. Yamamoto, "Error exponents of discrete memoryless channels and awgn channels with noisy feedback," in *2010 International Symposium On Information Theory and Its Applications*, Oct 2010, pp. 452–457.

[7] K. Palacio-Baus and N. Devroye, "Two-Way AWGN Channel Error Exponets at Zero-Rate," in *Proc. IEEE Int. Symp. Inf. Theory*, 2018.

[8] C. E. Shannon, "Two-way communications channels," in *4th Berkeley Symp. Math. Stat. Prob.*, Chicago, IL, Jun. 1961, pp. 611–644.

[9] M. Asadi and N. Devroye, "On the zero-error capacity of channels with noisy feedback," in *2017 55th Annual Allerton Conference on Communication, Control, and Computing*, Oct 2017, pp. 642–649.