

On Second Order Rate Regions for the Static Scalar Gaussian Broadcast Channel

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Abstract—This paper considers the single antenna, static, scalar Gaussian broadcast channel in the finite blocklength regime. Second order achievable and converse rate regions are presented. Both a global reliability and per-user reliability requirements are considered. The two-user case is analyzed in detail, and generalizations to the K -user case are also discussed.

The largest second order achievable regions presented here require both superposition and rate splitting in the code construction, as opposed to the (infinite blocklength, first order) capacity region which does not require rate splitting. Indeed, the finite blocklength penalty causes superposition alone to underperform other coding techniques in some parts of the region. In addition, the proposed scheme uses joint simultaneous decoding as opposed to successive interference cancellation.

Interestingly, in the two-user case with per-user reliability requirements, the capacity achieving superposition encoding order (with the codeword intended for the user with the smallest received SNR as cloud center) does not necessarily give the largest second order region. Instead, the message of the user with the smallest point-to-point second order capacity should be encoded in the cloud center in order to obtain the largest second order region for the proposed scheme.

Index Terms—URLLC; superposition coding; non-orthogonal multiple access; joint simultaneous decoding; rate splitting; finite blocklength; broadcast channel.

I. INTRODUCTION

Wireless communications is deeply integrated into many aspects of everyday life. The delivery on the promise of high bandwidth with reasonable latency has driven much interest into use cases that were previously considered less suitable for wireless communications. These are use cases requiring very low latency coupled with very high reliability. Wireless links are replacing wired links in remote, real-time control and monitoring in manufacturing, and in applications where wired links are impossible, such as unmanned aerial vehicles (UAV) and autonomous vehicles. For example, a key component of 5G New Radio, Ultra-Reliable and Low Latency Communications (URLLC) is the 5G service category with sub millisecond end-to-end delays and over 99.999% reliability [3] designed to meet these new requirements. Characterizing the performance of various code constructions operating under URLLC conditions has been a subject of interest [4], [5]. These works focus on an orthogonal URLLC operation, where communication is modeled as point-to-point links and makes uses of point-to-point results for channels at finite blocklength.

However, orthogonalization is known to lead to achievable rates below the capacity of many multi-user channels even in the infinite blocklength case. Thus, understanding the fundamental behavior of multi-user networks at finite blocklengths from an information theoretic standpoint is critical to benchmark various neXt URLLC generation (xURLLC) schemes.

In this paper, we derive approximations to the finite blocklength rate region for the single antenna, static, Gaussian broadcast channel in the spirit of the so-called *normal approximation* [6], which is a refined analysis of how the mutual information density concentrates to its mean as the blocklength increases while the error rate is kept fixed as the blocklength varies. The normal approximation quantifies how many bits can be sent through the channel within a finite number of channel uses while maintaining a given reliability.

Our proposed scheme uses rate splitting, superposition coding, and joint simultaneous decoding, which achieves the (infinite blocklength, or first order) capacity of the considered channel model [7, Section 5.2, and Problem 6.18]. When decoding for the two-user case, the user with the smallest receive SNR (referred to as the ‘weak user’) recovers its message while treating the message for the other user as noise. The weak user’s message is encoded into a codeword that is commonly referred to as the ‘cloud-center.’ The user with the largest receive SNR (referred to as the ‘strong user’) recovers jointly and simultaneously both messages, and its message is encoded into a codeword that is commonly referred to as the ‘satellite.’

While the benefits of superposition coding have been long known, the complexity of the decoder has meant that a consideration of the technique for practical networks has only emerged more recently [8]. Superposition coding has been included in many recent works on the benefits of non-orthogonal multiple access (NOMA) in future networks, see [9]–[11] and references therein. Most of the NOMA work has focused on superposition coding implemented with Successive Interference Cancellation (SIC) decoding [7, Figure 4.10]. SIC refers to the decoding process by which a user first decodes the messages intended to weaker users (i.e., users that have lower receive SNR), subtracts them from the received signal, and then decodes their own message. SIC is however not optimal in general, see for example the 3-user broadcast channel with degraded message sets in [7, Section 8.2]. Simultaneous (SIM) joint decoding, by which a user jointly decodes its own message and all the messages intended to weaker users, in general outperforms SIC in the infinite blocklength case [7, Problem 6.6]. SIM decoding has

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been considered in some recent NOMA literature, such as for example [12]. Our techniques presented here for SIM can be straightforwardly applied to SIC decoding as well.

Any NOMA scheme that requires some users to decode messages in addition to their own, opens up the possibility of rate-splitting [7, Proposition 4.1], by which the transmitter may allocate some portion of a message to be decoded by a set of users that are not otherwise interested in that message. Rate Splitting Multiple Access (RSMA) is a practical communication application of much recent interest (see [13] and references therein) that shows the use of rate splitting can outperform NOMA alone in certain practical cases. Here we investigate the second order properties of the most general version of superposition coding in [7, Problem 6.18], where it is shown that the alternative characterization of the capacity region for the more capable broadcast channel in [7, Section 5.6.1] can be obtained by using rate splitting, superposition coding, and Fourier–Motzkin elimination. *In other words, our proposed scheme uses rate splitting, superposition coding and SIM, and in doing so provides a possible information theoretic justification for the observed benefits seen by RSMA in the current literature.*

When considering finite blocklength operation of multi-user networks, care must be taken to how reliability is defined and measured. For the broadcast network, consisting of a single transmitter and multiple receivers, it can take two forms. It may be a *global* requirement of reliability, i.e., the joint probability of any user failing to decode its intended message, not exceeding a given value [7, Sec 5.1]. Alternatively, it may be a *per-user* requirement, where the probability of each user decoding their intended message(s) in error must not exceed a threshold specified for that user, which may differ across users. In xURLLC, some use cases will have varying reliability requirements and may be delivered simultaneously and in a non-orthogonal manner to other services, like enhanced Mobile Broadband (eMBB). Virtual/Augmented Reality applications will likely have relaxed reliability requirements compared to remote surgical applications. A transmitter that simultaneously sends entertainment information to one user while transmitting critical public safety information to another is another example. This network should not be constrained by a global error probability, as enforcing the most stringent reliability requirement may significantly reduce the overall performance. *This motivates us to consider both definitions of reliability in this work.*

Since the beginnings of information theory as a discipline, much effort has been spent in working to bridge between the elegant convergence of the optimal coding rate to capacity and results that give more practical insight. In short, what can be said about practical networks that operate at finite blocklength? The importance of these non-asymptotic fundamental limits to real networks was recognized very early and the first results were produced almost immediately by Shannon and Feinstein [14], [15], and then by Gallager [16]. In the ensuing years much progress was made in ‘large-deviation’ analysis, a study of the decay of the probability of increasingly unlikely events. This provided precise values for the rate of decay in the probability of error for fixed rates below capacity as

channel uses increased – the so-called ‘error exponent regime.’ Hayashi [17] and Polyanskiy *et al.* [6] improved the state of the art and derived tight non-asymptotic results for a variety of point-to-point channels assuming that the error probability remains fixed while the blocklength increases and the rate converges to capacity – the so-called ‘second order regime.’ *This work adopts the second order rate region perspective.*

The preceding discussion on non-asymptotic results concerned point-to-point communication problems. The practical usefulness of these results has driven significant interest in applying similar techniques to multi-user channels. Much work has focused on the Multiple Access Channels (MAC), such as [18]–[20], which considered both the discrete memoryless and the AWGN models. Interestingly, for the Gaussian MAC the second order region is not tightly characterized yet. Other variations on the MAC at finite blocklength have been considered – such as, fading and random access [21], the number of users scales with the blocklength [22], [23], with output feedback [24], with cooperation [25], etc. – but those are not directly relevant to this work. Directly relevant to our work is [26], which considered the Gaussian MAC *with degraded message sets*, that is, one of the two transmitters knows both messages at the time of encoding; in this case the second order region is known. In our conference paper [1], we made use of several techniques developed in [20], [26], such as the multivariate Berry-Essen Theorem and methods for bounding the probability of error for threshold decoding, which we extend here to the case of any number of users and also to the case of per-user reliabilities.

The Broadcast channel (BC) at finite blocklength has been studied for example in [19], where an achievable region for the two-user, discrete memoryless, *asymmetric* (where one receiver has to decode both messages) BC was presented. In [27], the two-user AWGN BC with heterogeneous blocklengths was considered; our work with global error is the special case where the two blocklengths are the same, yet our construction produces a larger region in this case. In [28], [29] the AWGN BC channel with superposition coding was analyzed based on point-to-point results; it is unclear which code construction would achieve the dispersion utilized in the analysis, possibly that in [30].

Many second order results, including our own, rely on power-shell codebook construction. A power shell for a codebook of length n is the $(n-1)$ sphere centered at zero whose radius is \sqrt{nP} , where P is the average input power constraint. A power shell construction is a random coding argument where codewords are chosen uniformly at random from that $(n-1)$ -sphere. Power shell construction aligns with Shannon’s observation about the optimal decay of the probability of error near capacity of the point-to-point Gaussian channel, which is achieved by codewords on the power-shell [31].

A. Contributions

In this paper we aim to characterize the second order rate region of the K -user single antenna, static, Gaussian BC, under global and per-user reliability constraints, in the case where the users have the same blocklength. Our main contributions are as follows.

- **(1) Achievability.** By utilizing modified techniques from [26], we show that superposition coding with rate splitting provides the largest second order achievable rate region for this BC network in the case of two users. Through the addition of rate splitting, our achievable region for the two-user case is a super-set of the region presented in [27] evaluated for equal blocklength for the users. An extension to any number of users, albeit without rate splitting, is also given.
- **(2) Converse.** We generalize the converse argument provided in [27] to the K -user case, as well as to the per-user reliability constraints, which to the best of our knowledge has never been reported before.
- **(3) Unexpected behavior under per-user error.** Finally, for the case of per-user reliability and two users, we show that the capacity achieving ordering of superposition coding, where the weak user's message is encoded in the cloud center, and the strong user's message is superimposed as a satellite, does not always achieve the largest second order region. The optimal ordering is instead determined by the second order point-to-point capacities between the transmitter and each of the users. For strictly more than two users, the best superposition coding order with per-user reliabilities changes for different points on the boundary of the second order region.

Several natural questions arise from our analysis. The first concerns how our results change when the channel is subject to a random fading process, and closely related is the performance for multiple antenna BC. Our results hold for the SIMO AWGN case where the ordering is performed using the post receive combining effective scalar channel's SNR. Multiple antennas at the transmitter and fading in general lead to complex optimization problems. For point to point channels recent advancements have been made in [32], [33] that point the way to extending this analysis. Another question is how far from these theoretical limits are practical codes. As shown in [34, Sec 5.2], there is still a sizeable difference between rates achieved by modern codes and what has been shown to be achievable theoretically for P2P AWGN channels, especially for shorter blocklengths. Closing this gap for P2P channels remains an active area of research, as does the development of superimposed codes that approach the theoretically achievable bounds.

B. Notation

For reals $a \leq b$, we let $[a, b] := \{x : a \leq x \leq b\}$. For integers $a \leq b$, we let $[a : b] := \{a, a+1, \dots, b\}$ and $[b] := [1 : b]$. $\delta(\cdot)$ is the delta Dirac function. We write $f(x) = O(g(x))$ if a positive M and an x_0 can be found such that $|f(x)| \leq Mg(x)$ for all $x \geq x_0$; we also use O_n as a shorthand notation for $O(n)$. We refer to real-valued vectors of length n as \mathbf{x} (bold font). We include a superscript (x^n) when we wish to highlight the length of the vector. $\mathbf{1}$ and $\mathbf{0}$ denote the all-one and all-zero vector (or matrix), respectively. For vectors \mathbf{a} and \mathbf{b} in \mathbb{R}^n , the inner product is denoted as $\langle \mathbf{a}, \mathbf{b} \rangle = \sum_{i \in [n]} a_i b_i$,

which induces the norm $\|\mathbf{a}\| = \sqrt{\langle \mathbf{a}, \mathbf{a} \rangle}$. The $(n-1)$ -sphere of radius $r > 0$ is the set

$$\mathcal{S}_{n-1}(r) = \{\mathbf{a} \in \mathbb{R}^n : \|\mathbf{a}\| = \sqrt{r}\}, \quad (1)$$

whose surface area is denoted as

$$S_n(r) = \frac{2\pi^{n/2}}{\Gamma(n/2)} r^{n-1}. \quad (2)$$

Note that the set in (1) is denoted by the calligraphic font and has subscript $n-1$, while the real non-negative number in (2) is denoted by the normal font and has subscript n as in [26].

$\mathbf{Z} \sim \mathcal{N}(\boldsymbol{\mu}, \mathbf{V})$ denotes that \mathbf{Z} is a jointly Gaussian vector with mean $\boldsymbol{\mu}$ and covariance matrix \mathbf{V} , with cumulative distribution function (cdf)

$$\Psi(\mathbf{x}; \boldsymbol{\mu}, \mathbf{V}) = \Pr[\mathbf{Z} \leq \mathbf{x}], \quad (3)$$

where the inequality " $\mathbf{Z} \leq \mathbf{x}$ " in (3) is intended component-wise, and with probability distribution function (pdf)

$$\mathcal{N}(\mathbf{x}; \boldsymbol{\mu}, \mathbf{V}) = \frac{\partial \Psi(\mathbf{x}; \boldsymbol{\mu}, \mathbf{V})}{\partial \mathbf{x}} = \frac{e^{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T \mathbf{V}^{-1}(\mathbf{x}-\boldsymbol{\mu})}}{\sqrt{\det[2\pi \mathbf{V}]}}. \quad (4)$$

Following [26, eq(33)], for $\varepsilon \in [0, 1]$ and covariance matrix \mathbf{V} , we define the set

$$\mathcal{Q}_{\text{inv}}(\varepsilon; \mathbf{V}) = \{\mathbf{a} : \Psi(-\mathbf{a}; \mathbf{0}, \mathbf{V}) \geq 1 - \varepsilon\}. \quad (5)$$

The *capacity*, in nats per channel use, of the point-to-point Gaussian channel with SNR x is

$$\mathcal{C}(x) = 1/2 \ln(1+x), \quad 0 \leq x. \quad (6)$$

Second order results for multi-user Gaussian channels are often expressed as a function of the *cross-dispersion* function

$$\mathcal{V}(x, y) = \frac{x(2+y)}{2(1+x)(1+y)}, \quad 0 \leq x \leq y. \quad (7)$$

The point-to-point Gaussian *dispersion* function is

$$\mathcal{V}(x) = \mathcal{V}(x, x) = \frac{x(2+x)}{2(1+x)^2}, \quad 0 \leq x. \quad (8)$$

The *normal approximation* of the second order capacity of the point-to-point Gaussian channel with SNR x , for n channel uses and reliability ε , is denoted as

$$\kappa(n, x, \varepsilon) = \mathcal{C}(x) - \sqrt{\frac{\mathcal{V}(x)}{n}} \mathcal{Q}^{-1}(\varepsilon), \quad 0 \leq x, \varepsilon \in [0, 1], \quad (9)$$

which is an accurate proxy for achievable rates for values of the parameters for which $\kappa(n, \gamma, \varepsilon)$ is at least comparable with $\ln(n)/n$ [6]. In (9), $\mathcal{Q}^{-1}(\cdot)$ denotes the inverse of the function

$$\mathcal{Q}(x) = \int_x^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt, \quad x \in \mathbb{R}. \quad (10)$$

For the scalar case, the set defined in (5) is

$$\mathcal{Q}_{\text{inv}}(\varepsilon; \sigma^2) = \{a \in \mathbb{R} : a \leq -\sqrt{\sigma^2} \mathcal{Q}^{-1}(\varepsilon)\}, \quad \varepsilon \in [0, 1]. \quad (11)$$

The set in (11) only contains negative values for $\varepsilon \in [0, 1/2)$.

II. PROBLEM FORMULATION

We consider the memoryless K -user real-valued static Additive White Gaussian Noise (AWGN) Broadcast Channel (BC), where the channel between the base-station sending signal X and the multiple receivers is modeled as $Y_i = h_i X + Z_i$ for user $i \in [K]$. Here h_i is the constant, real-valued channel state between transmitter and receiver i , and Z_i is the Gaussian noise at receiver i , assumed to be independent of all other noises and of the input, and have zero mean and variance σ_i^2 . The input X is subject to the power constraint $\mathbb{E}[X^2] \leq P$. Given these normalizations, the SNR at receiver i is $\gamma_i := P|h_i|^2/\sigma_i^2$, $i \in [K]$.

We are interested in the so-called *second order regime*, where the block-length n is assumed to be large, but not infinite, and the average probability of error is bounded by ε , which may be small but not vanishing in n . For most memoryless point-to-point channels, it has been shown [6], [17] that $M^*(n, \varepsilon)$, defined as the maximum number of messages that can be sent within n channel uses and with an average probability of error not exceeding ε , behaves as

$$1/n \ln M^*(n, \varepsilon) = \kappa(n, \gamma, \varepsilon) + O_{\ln(n)/n}, \quad (12)$$

where the normal approximation function $\kappa(\cdot)$ was defined in (9), and where the term $\sqrt{V(\gamma)/n} Q^{-1}(\varepsilon)$ concisely captures the rate penalty incurred by forcing decoding after n channel uses and allowing a probability of error no larger than $\varepsilon \in (0, 1)$ on a point-to-point Gaussian channel with SNR γ . In this paper we aim to develop expressions akin to (12) for the two-user AWGN BC. We will also provide extensions to any number of users. We start with the formal definition of the second order region for the two-user case, which can be straightforwardly extended to any number of users.

Definition 1 (Code with Global Error). Given integer sets $(\mathcal{M}_0, \mathcal{M}_1, \mathcal{M}_2)$, integer n , and non-negative reals (P, ϵ) , an $(n, |\mathcal{M}_0|, |\mathcal{M}_1|, |\mathcal{M}_2|, P, \epsilon)$ code for the two-user AWGN BC has: (i) three independent and uniformly distributed messages on $\mathcal{M}_0 \times \mathcal{M}_1 \times \mathcal{M}_2$; (ii) one encoder function $\text{enc} : \mathcal{M}_0 \times \mathcal{M}_1 \times \mathcal{M}_2 \rightarrow \mathbb{R}^n$ with power constraint

$$\|\text{enc}(m_0, m_1, m_2)\|^2 \leq nP, \quad (13)$$

for all $(m_0, m_1, m_2) \in \mathcal{M}_0 \times \mathcal{M}_1 \times \mathcal{M}_2$; and (iii) two decoder functions $\text{dec}_k : \mathbb{R}^n \rightarrow \mathcal{M}_0 \times \mathcal{M}_k$, $k \in [2]$, with average global probability of error satisfying

$$\Pr \left[\bigcup_{k \in [2]} \text{dec}_k(\mathbf{Y}_k^n) \neq (W_0, W_k) \right] \leq \varepsilon, \quad (14)$$

where in (14) it is understood that (W_0, W_1, W_2) was sent. \square

We shall use ε to denote the largest allowed average probability of error, and ϵ_n for the probability of error of a code of block-length n . Again note the difference in font type.

Definition 2 (Second Order Capacity Region with Global Error). A non-negative rate tuple (R_0, R_1, R_2) is said to be (n, ε) -achievable if there exists a $(n, M_{0,n}, M_{1,n}, M_{2,n}, P, \epsilon_n)$ code with global error for some n with $\epsilon_n \leq \varepsilon$ and $\frac{\ln(M_{j,n})}{n} \geq R_j$ for $j \in \{0, 1, 2\}$. Let $\mathcal{C}(n, \varepsilon)$ denote the set of all (n, ε) -achievable rate tuples, referred to as the *second order capacity region (with global error)*. \square

Definition 3 (Capacity Region). The capacity region \mathcal{C} is

$$\mathcal{C}(\varepsilon) = \bigcup_{n \geq 1} \mathcal{C}(n, \varepsilon), \quad (\varepsilon\text{-capacity region}), \quad (15)$$

$$\mathcal{C} = \bigcap_{\varepsilon > 0} \mathcal{C}(\varepsilon), \quad (\text{capacity region}). \quad (16)$$

The two-user Gaussian BC enjoys a strong converse [35], that is, the capacity region satisfies (where WLOG $\gamma_1 \geq \gamma_2$)

$$\mathcal{C} = \mathcal{C}(\varepsilon) = \bigcup_{\alpha \in [0, 1]} \left\{ (R_0, R_1, R_2) \in \mathbb{R}_+^3 : \right. \quad (17a)$$

$$R_0 + R_2 \leq \mathcal{C} \left(\frac{(1 - \alpha)\gamma_2}{1 + \alpha\gamma_2} \right), \quad (17b)$$

$$\left. R_1 \leq \mathcal{C}(\alpha\gamma_1) \right\}. \quad (17c)$$

where α is interpreted as the power split parameter. \square

Goal. We aim to find, or bound, the second order region $\mathcal{C}(n, \varepsilon)$ by characterizing the rate penalty terms to be included in the capacity region in (17) akin to the term $\sqrt{V(\gamma)/n} Q^{-1}(\varepsilon)$ in (9) for point-to-point channels.

Remark 1 (On Per-User Error). We shall also use, instead of the global probability of error in (14), the per-user average error probability criteria

$$\Pr[\text{dec}_k(Y_k^n) \neq (W_0, W_k)] \leq \varepsilon_k, \quad k \in [2]. \quad (18)$$

The definition of code and second order region with per-user error in (18) follow similarly to those with global error and is not repeated here for sake of space. \square

III. MAIN RESULT

The main result of this paper for the two-user case is summarized in Theorem 1. The converse proof can be found in Section V and the achievability in Section VI, with all the details regarding rate splitting, superposition coding and SIM decoding. Extensions to the K -user case can be found in Sections V-A and VI-A.

Theorem 1 (Second Order Regions with Global Error). *Given the model in Section II for global error ε , we have*

$$\mathcal{R}^{(\text{SUP})}(n, \varepsilon) \subseteq \mathcal{C}(n, \varepsilon) \subseteq \mathcal{R}^{(\text{CS})}(n, \varepsilon), \quad (19)$$

where the regions $\mathcal{R}^{(\text{SUP})}(n, \varepsilon)$ and $\mathcal{R}^{(\text{CS})}(n, \varepsilon)$ are as follows.

The region $\mathcal{R}^{(\text{SUP})}(n, \varepsilon)$ is attained by rate splitting, superposition coding, and SIM, and is given by

$$\begin{aligned} \mathcal{R}^{(\text{SUP})}(n, \varepsilon) = \bigcup_{(\alpha, \beta, \epsilon_{10}, \epsilon_{11}, \epsilon_2) \in [0, 1]^5} \left\{ (R_0, R_1, R_2) \in \mathbb{R}_+^3 : \right. \\ R_0 + R_2 + \beta R_1 \leq \mathcal{C} \left(\frac{(1 - \alpha)\gamma_2}{1 + \alpha\gamma_2} \right) \\ \left. - \sqrt{\frac{1}{n}} V'(\alpha\gamma_2, \gamma_2) Q^{-1}(\epsilon_2) + O_{\ln(n)/n}, \right. \quad (20a) \\ (1 - \beta)R_1 \leq \kappa(n, \alpha\gamma_1, \epsilon_{10}) + O_{\ln(n)/n}, \quad (20b) \\ \left. R_0 + R_1 + R_2 \leq \kappa(n, \gamma_1, \epsilon_{11}) + O_{\ln(n)/n} \right\}, \quad (20c) \end{aligned}$$

where α is the power split and β the rate split. The dispersion in (20a) is defined as

$$V'(\alpha\gamma_2, \gamma_2) := V(\alpha\gamma_2) + V(\gamma_2) - 2V(\alpha\gamma_2, \gamma_2) \quad (21a)$$

$$= \frac{(1-\alpha)\gamma_2(2\alpha\gamma_2^2 + \gamma_2 + 3\alpha\gamma_2 + 2)}{2(\gamma_2 + 1)^2(\alpha\gamma_2 + 1)^2}, \quad (21b)$$

with $V(\cdot, \cdot)$ and $V(\cdot)$ are defined in (7) and (8), respectively. The triplet $(\epsilon_{10}, \epsilon_{11}, \epsilon_2) \in [0, 1]^3$ satisfies

$$(1 - \epsilon_1)(1 - \epsilon_2) \geq 1 - \epsilon, \quad (22)$$

where ϵ_1 is the error rate at receiver 1 which satisfies

$$F(\epsilon_{10}, \epsilon_{11}; r(\alpha\gamma_1, \gamma_1)) \geq 1 - \epsilon, \quad (23)$$

where the probability of correct decoding function $F(\cdot, \cdot; \cdot)$ is

$$F(\epsilon_{10}, \epsilon_{11}; r) := \Pr[G_2 \leq Q^{-1}(\epsilon_{10}), \quad (24a)$$

$$rG_2 + \sqrt{1-r^2}G_3 \leq Q^{-1}(\epsilon_{11})], \quad (24b)$$

for G_2, G_3 i.i.d. standard Gaussian random variables, and the correlation coefficient $r(\alpha\gamma_1, \gamma_1)$ in (23) is defined as

$$r(\alpha\gamma_1, \gamma_1) := \frac{V(\alpha\gamma_1, \gamma_1)}{\sqrt{V(\alpha\gamma_1)V(\gamma_1)}} = \sqrt{\frac{(2+\gamma_1)\alpha}{(2+\alpha\gamma_1)}}. \quad (25)$$

The region $\mathcal{R}^{(CS)}(n, \epsilon)$ is the cut-set-type region

$$\mathcal{R}^{(CS)}(n, \epsilon) = \left\{ (R_0, R_1, R_2) \in \mathbb{R}_+^3 : \quad (26a)$$

$$R_0 + R_1 \leq \kappa(n, \gamma_1, \epsilon) + O_{\ln(n)/n}, \quad (26b)$$

$$R_0 + R_2 \leq \kappa(n, \gamma_2, \epsilon) + O_{\ln(n)/n}, \quad (26c)$$

$$R_0 + R_1 + R_2 \leq \kappa(n, \max(\gamma_1, \gamma_2), 2\epsilon) + O_{\ln(n)/n} \}. \quad (26d)$$

Remark 2 (Second Order Regions with Per-User Error). In Theorem 1, the achievable second order region $\mathcal{R}^{(SUP)}(n, \epsilon)$ in (20) without the constraint in (22), which links the error rates at the two receivers (that experience independent noise by assumption), gives an achievable region for the case with per-user error criteria. When we remove the constraint in (22), we indicate the achievable region as $\mathcal{R}^{(SUP)}(n, \epsilon_1, \epsilon_2)$ to stress the two per-user probability of error requirements.

With per-user error, the achievable region akin to the one in Theorem 1 is $\mathcal{R}^{(SUP1)}(n, \epsilon_1, \epsilon_2) \cup \mathcal{R}^{(SUP2)}(n, \epsilon_1, \epsilon_2)$, where $\mathcal{R}^{(SUP2)}(n, \epsilon_1, \epsilon_2)$ is the region in (20) (with the superposition coding order that is capacity achieving under the assumption $\gamma_1 \geq \gamma_2$), and the region $\mathcal{R}^{(SUP1)}(n, \epsilon_1, \epsilon_2)$ is similar to the region in (20) but with the role of the users swapped (that is, with the message of user 1 in the cloud center). While swapping the order of superposition coding does not appear to enlarge the achievable region in Theorem 1 for global error, it provides improvements when one considers per-user error as we will show in Section IV.

The outer bound region $\mathcal{R}^{(CS)}(n, \epsilon)$ in Theorem 1 can also be extended to the case of per-user error. In particular, the single user bounds read $R_0 + R_j \leq \kappa(n, \gamma_j, \epsilon_j) + O_{\ln(n)/n}$ for $j \in [2]$, and the sum-rate bound becomes $R_0 + R_1 + R_2 \leq \kappa(n, \max(\gamma_1, \gamma_2), \epsilon_1 + \epsilon_2) + O_{\ln(n)/n}$. \square

Remark 3 (On the Dispersion of Decoding the Message in the Cloud Center). Let

$$x := \alpha\gamma_2 \leq y := \gamma_2, \quad z := \frac{y-x}{1+x} = \frac{(1-\alpha)\gamma_2}{1+\alpha\gamma_2}, \quad (27)$$

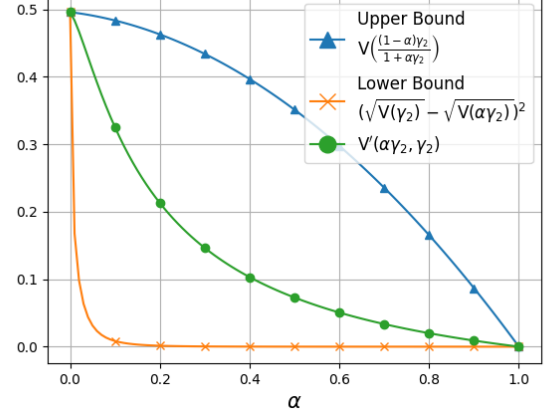


Fig. 1: Dispersions vs. α for $\gamma_2 = 10$.

where z represents the SINR in decoding the cloud center by treating the satellite as a noise in Theorem 1. The dispersion $V'(\cdot, \cdot)$ in (21a) can be upper bounded as follows

$$V'(x, y) = V(x) + V(y) - 2V(x, y) \quad (28a)$$

$$= \frac{(y-x)(2xy + 3x + y + 2)}{2(1+x)^2(1+y)^2} \quad (28b)$$

$$= \frac{z(2 + \frac{2x+1}{x+1})}{2(1+z)^2(1+x)} \leq V(z), \quad (28c)$$

and lower bounded as follows

$$V'(x, y) \geq V(x) + V(y) - 2\sqrt{V(x)V(y)} \quad (28d)$$

$$= (\sqrt{V(y)} - \sqrt{V(x)})^2. \quad (28e)$$

Recall that $V'(\cdot, \cdot)$ in (21a) is the dispersion for the rate of messages carried by the cloud center. From the upper bound in (28c), we see that $V'(\cdot, \cdot)$ in our scheme is lower than the dispersion of a point-to-point Gaussian channel in which the interference from the satellite codeword is treated as Gaussian noise. We do not have at present an intuitive interpretation of the lower bound in (28e). The dispersion $V'(\alpha\gamma, \gamma)$ vs. α is depicted in Fig. 1. In [30, Theorem 2] the Authors considered the performance of nearest-neighbor decoding of independent codewords drawn uniformly at random from two classes of distributions. We note that $V'(\cdot, \cdot)$ in (21a) is the special case of [30, Eq(23)] for codes on the power sphere for the AWGN channel with two users. The same paper also shows that with i.i.d. Gaussian codes, on the AWGN channel, and with nearest-neighbor decoding, the dispersion is [30, Eq(27)], which equals $z/(1+z)$ where z is the SINR. The dispersion $z/(1+z)$ is often used to assess NOMA performance by means of (sub-optimal) point-to-point results. \square

Remark 4 (On Reliability Allocation). The probability of correct decoding function in (24) is monotonic in the correlation coefficient $r \in [-1, 1]$. Some of its values are

$$F(\epsilon_0, \epsilon_1; +1) = \Pr[G_2 \leq Q^{-1}(\epsilon_0), G_2 \leq Q^{-1}(\epsilon_1)] \quad (29a)$$

$$= \Pr[G_2 \leq \min(Q^{-1}(\epsilon_0), Q^{-1}(\epsilon_1))] \quad (29b)$$

$$= 1 - \max(\epsilon_0, \epsilon_1); \quad (29c)$$

and

$$F(\epsilon_0, \epsilon_1; 0) = \Pr[G_2 \leq Q^{-1}(\epsilon_0), G_3 \leq Q^{-1}(\epsilon_1)] \quad (30a)$$

$$= \Pr[G_2 \leq Q^{-1}(\epsilon_0)] \Pr[G_3 \leq Q^{-1}(\epsilon_1)] \quad (30b)$$

$$= (1 - \epsilon_0)(1 - \epsilon_1); \quad (30c)$$

and

$$F(\epsilon_0, \epsilon_1; -1) = \Pr[G_2 \leq Q^{-1}(\epsilon_0), -G_2 \leq Q^{-1}(\epsilon_1)] \quad (31a)$$

$$= \Pr[Q^{-1}(1 - \epsilon_1) \leq G_2 \leq Q^{-1}(\epsilon_0)] 1_{\{1 - \epsilon_1 \geq \epsilon_0\}} \quad (31b)$$

$$= [1 - \epsilon_1 - \epsilon_0]^+. \quad (31c)$$

We thus conclude that the “error rates region” $\{(\epsilon_0, \epsilon_1) \in [0, 1]^2 : F(\epsilon_0, \epsilon_1; r) \geq 1 - \varepsilon\}$ monotonically enlarges with r from the triangle $\epsilon_0 + \epsilon_1 \leq \varepsilon$ for $r = -1$, to the square $\max(\epsilon_0, \epsilon_1) \leq \varepsilon$ for $r = +1$, as depicted in Fig. 2. This is the set of reliability pairs that we can optimize over in the superposition coding inner bound for receiver 1. Indeed, consider the function $1 - F(\epsilon_{\text{sat}}, \epsilon_{\text{cc}}; r) = \epsilon_1$, which is the average probability of error at receiver 1. It includes two terms: ϵ_{sat} is related to the reliability of decoding the satellite codeword, and ϵ_{cc} is related to the probability of erroneous decoding of the cloud center codeword (and thus also the satellite). A decoding event in which the receiver recovers the satellite, but fails to recover the cloud center would be counted towards ϵ_{cc} , but not ϵ_{sat} . Overall, the optimization in the superposition coding achievable region implies that we can choose the best reliability allocation among these two terms in order to achieve an overall reliability ϵ_1 at receiver 1.

As this optimization concerns a single user, it is relevant to both the global and per-user reliability cases. For global error, a further optimization step in the achievable region is possible: we can choose overall reliability ϵ_1 at receiver 1 and ϵ_2 at receiver 2 such that $1 - (1 - \epsilon_1)(1 - \epsilon_2) \leq \varepsilon$, where ε is the maximum global average probability of error. Therefore, we see that reliability optimization can be leveraged to optimize the downlink performance with latency constraints. \square

Remark 5 (On Time Division with Global Error). A baseline scheme for the case of private rates only, that is, for $R_0 = 0$, is the second order region achieved by Time Division Multiplexing (TDM) with power control, given by

$$\mathcal{R}^{(\text{TDM})}(n, \varepsilon) = \bigcup_{\substack{(\tau_1, \tau_2, \epsilon_1, \epsilon_2) \in [0, 1]^4, (\alpha_1, \alpha_2) \in \mathbb{R}_+^2: \\ \tau_1 + \tau_2 \leq 1, \tau_1 \alpha_1 + \tau_2 \alpha_2 \leq 1 \\ (1 - \epsilon_1)(1 - \epsilon_2) \geq 1 - \varepsilon}} \{(R_1, R_2) \in \mathbb{R}_+^2 : \quad (32a)$$

$$R_1 \leq \tau_1 \kappa(\tau_1 n, \alpha_1 \gamma_1, \epsilon_1) + O(\ln(\tau_1 n)/n), \quad (32b)$$

$$R_2 \leq \tau_2 \kappa(\tau_2 n, \alpha_2 \gamma_2, \epsilon_2) + O(\ln(\tau_2 n)/n)\}, \quad (32c)$$

where $\tau_j n$ channel uses are allocated to receiver j , subject to the total time constraint $\tau_1 + \tau_2 \leq 1$; where power $\alpha_j P$ is allocated to receiver j , subject to the average power constraint $\tau_1 \alpha_1 + \tau_2 \alpha_2 \leq 1$; and where ϵ_j is the reliability allocated to receiver j , subject to the average probability of error constraint $(1 - \epsilon_1)(1 - \epsilon_2) \geq 1 - \varepsilon$ (as the noises are assumed to be independent). We shall plot this region in our numerical evaluations. Numerically we observed that α_2 is always greater

than α_1 for points on the boundary of $\mathcal{R}^{(\text{TDM})}(n, \varepsilon)$, however the optimal parameters are difficult to describe analytically as they are linked with the optimization of the time split parameters τ_1, τ_2 and of the reliabilities ϵ_1, ϵ_2 . \square

Remark 6 (On Concatenate & Code with Global Error). The choice $\beta = 1$ in $\mathcal{R}^{(\text{SUP})}(n, \varepsilon)$ means that no satellite codewords are sent, that is both users decode the same codeword with each user recovering their message from some fraction of the bits encoded. In [1] we referred to this case as *Concatenate & Code Protocol* (CCP). CCP is obtained as a special case of $\mathcal{R}^{(\text{SUP})}(n, \varepsilon)$ for $\alpha = 0$ and $\epsilon_{10} = 0$, resulting in $V'(0, \gamma_2) = V(\gamma_2)$. Thus, the CCP region is

$$\begin{aligned} \mathcal{R}^{(\text{CCP})}(n, \varepsilon) &= \{(R_0, R_1, R_2) \in \mathbb{R}_+^3 : R_0 + R_1 + R_2 \leq \max_{\substack{(\epsilon_1, \epsilon_2) \in [0, 1]^2 \\ (1 - \epsilon_2)(1 - \epsilon_1) \geq 1 - \varepsilon}} \min(\kappa(n, \gamma_1, \epsilon_1), \kappa(n, \gamma_2, \epsilon_2)) + O_{\ln(n)/n}\}. \end{aligned} \quad (33a)$$

We note that it is possible to have $\kappa(n, \gamma_1, \epsilon_1) < \kappa(n, \gamma_2, \epsilon_2)$ even under the assumption $\gamma_1 > \gamma_2$ if $\epsilon_1 \ll \epsilon_2$. Numerically we observed that the optimal reliability allocation is $\epsilon_1 \leq \epsilon_2 \approx \varepsilon$ such that $\kappa(n, \gamma_1, \epsilon_1) = \kappa(n, \gamma_2, \epsilon_2)$. \square

Remark 7 (On Superposition Coding without Rate Splitting with Global Error). An achievable region without rate splitting is obtained by setting $\beta = 0$ in $\mathcal{R}^{(\text{SUP})}(n, \varepsilon)$; in this case we numerically observed that the sum-rate bound is always tight (that is, $\text{eq}(20a) + \text{eq}(20b) = \text{eq}(20c)$), and that the optimal reliability allocation is such that $\epsilon_{11} \ll \epsilon_{10} \approx \epsilon_1$. We shall refer to this region as $\mathcal{R}^{(\text{SUPNoRS})}(n, \varepsilon)$, given by

$$\mathcal{R}^{(\text{SUPNoRS})}(n, \varepsilon) = \bigcup_{\substack{(\alpha, \epsilon_2, \epsilon_1) \in [0, 1]^3 \\ \text{eq}(20a) + \text{eq}(20b) = \text{eq}(20c) \\ (1 - \epsilon_2)(1 - \epsilon_1) \geq 1 - \varepsilon}} \{(R_0, R_1, R_2) \in \mathbb{R}_+^3 : \quad (34a)$$

$$R_0 + R_2 \leq C \left(\frac{(1 - \alpha)\gamma_2}{1 + \alpha\gamma_2} \right) \quad (34b)$$

$$- \sqrt{\frac{1}{n}} V'(\alpha\gamma_2, \gamma_2) Q^{-1}(\epsilon_2) + O_{\ln(n)/n}, \quad (34c)$$

$$R_1 \leq C(\alpha\gamma_1) - \sqrt{\frac{1}{n}} V(\alpha\gamma_1) Q^{-1}(\epsilon_1) + O_{\ln(n)/n}\}. \quad (34d)$$

Numerically we also observed that optimizing $\beta \in [0, 1]$ in $\mathcal{R}^{(\text{SUP})}(n, \varepsilon)$ always results in either $\mathcal{R}^{(\text{CCP})}(n, \varepsilon)$ or $\mathcal{R}^{(\text{SUPNoRS})}(n, \varepsilon)$. \square

Remark 8 (On SIM vs SIC with Global Error).

$$\mathcal{R}^{(\text{SUP-SIC})}(n, \varepsilon) = \bigcup_{(\alpha, \beta, \epsilon_{10}, \epsilon_{11}, \epsilon_2) \in [0, 1]^5} \{(R_0, R_1, R_2) \in \mathbb{R}_+^3 : \quad (35a)$$

$$R_0 + R_2 + \beta R_1 \leq C \left(\frac{(1 - \alpha)\gamma_2}{1 + \alpha\gamma_2} \right) - \sqrt{\frac{1}{n}} V'(\alpha\gamma_2, \gamma_2) Q^{-1}(\epsilon_2) + O_{\ln(n)/n}, \quad (35b)$$

$$\begin{aligned} (1 - \beta)R_1 &\leq \kappa(n, \alpha\gamma_1, \epsilon_{10}) + O_{\ln(n)/n}, \\ R_0 + R_2 + \beta R_1 &\leq C \left(\frac{(1 - \alpha)\gamma_1}{1 + \alpha\gamma_1} \right) - \sqrt{\frac{1}{n}} V(\alpha\gamma_1, \gamma_1) Q^{-1}(\epsilon_{11}) + O_{\ln(n)/n}, \end{aligned} \quad (35c)$$

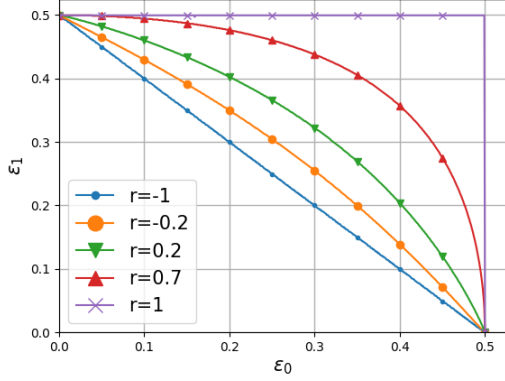


Fig. 2: Region $\{(\epsilon_0, \epsilon_1) \in [0, 1]^2 : F(\epsilon_0, \epsilon_1; r) \geq 1 - \varepsilon = 0.5\}$ for various values of r .

where α is the power split and β the rate split. The dispersion in (35a) and (35c) is defined as in (21a) with $V(\cdot, \cdot)$ and $V(\cdot)$ are defined in (7) and (8), respectively. The triplet $(\epsilon_{10}, \epsilon_{11}, \epsilon_2) \in [0, 1]^3$ satisfies

$$(1 - \epsilon_1)(1 - \epsilon_2) \geq 1 - \varepsilon, \quad (37)$$

where ϵ_1 is the error rate at receiver 1 which satisfies

$$F(\epsilon_{10}, \epsilon_{11}; r(\alpha\gamma_1, \gamma_1)) \geq 1 - \epsilon_1, \quad (38)$$

where the probability of correct decoding function $F(\cdot, \cdot; \cdot)$ is

$$F(\epsilon_{10}, \epsilon_{11}; r) := \Pr [G_2 \leq Q^{-1}(\epsilon_{10}), \quad (39a)$$

$$rG_2 + \sqrt{1 - r^2}G_3 \leq Q^{-1}(\epsilon_{11})], \quad (39b)$$

for G_2, G_3 i.i.d. standard Gaussian random variables, and the correlation coefficient $r(\alpha\gamma_1, \gamma_1)$ in (38) is defined as

$$r = \frac{V(\alpha\gamma_1, \gamma_1) - V(\alpha\gamma_1, \alpha\gamma_1)}{\sqrt{(V(\gamma_1, \gamma_1) + V(\alpha\gamma_1, \alpha\gamma_1) - 2V(\alpha\gamma_1, \gamma_1))V(\alpha\gamma_1, \alpha\gamma_1)}}. \quad (40)$$

Extensive numerical simulations show that $\mathcal{R}^{(\text{SUP+SIC})}(n, \varepsilon)$ and $\mathcal{R}^{(\text{SUP})}(n, \varepsilon)$, which uses SIM, are equivalent to a limit an order of magnitude less than the $O_{\ln(n)/n}$ higher order terms of the approximation. \square

Remark 9 (Extension of TDM, CCP, SUPnoRS, SUP+SIC, to the case of per-user error). One can define regions $\mathcal{R}^{(\text{TDM})}(n, \epsilon_1, \epsilon_2)$ (akin to (32)), $\mathcal{R}^{(\text{CCP})}(n, \epsilon_1, \epsilon_2)$ (akin to (33)), $\mathcal{R}^{(\text{SUPnoRS})}(n, \epsilon_1, \epsilon_2)$ (akin to (34)), and $\mathcal{R}^{(\text{SUP+SIC})}(n, \epsilon_1, \epsilon_2)$ (akin to (35)) for per-user error by removing the constraint $(1 - \epsilon_2)(1 - \epsilon_1) \geq 1 - \varepsilon$ in the respective optimizations. The order of superposition can also be swapped in order to possibly obtain larger achievable regions. \square

IV. NUMERICAL EVALUATIONS

We start by giving numerical evaluations of the second order rate region in Theorem 1 for private rates only, that is, $R_0 = 0$. Numerically we observed that: (i) achievable regions are not convex when the normal approximation terms become

comparable to $\ln(n)/n$, which is the areas highlighted in grey in the figures; and (ii) $\beta \in (0, 1)$ never gives a point on the boundary of the region $\mathcal{R}^{(\text{SUP})}(n, \varepsilon)$, that is, $\mathcal{R}^{(\text{SUP})}(n, \varepsilon)$ is the union of $\mathcal{R}^{(\text{SUPnoRS})}(n, \varepsilon)$ in (34) and $\mathcal{R}^{(\text{CCP})}(n, \varepsilon)$ in (33).

In Fig. 3 we plot in the left column the regions $\mathcal{R}^{(\text{SUPnoRS})}(n, \varepsilon)$, $\mathcal{R}^{(\text{CCP})}(n, \varepsilon)$, and $\mathcal{R}^{(\text{SUP})}(n, \varepsilon)$. As a baseline, we plot $\mathcal{R}^{(\text{TDM})}(n, \varepsilon)$ in (32). As a converse bound, we plot $\mathcal{R}^{(\text{CS})}(n, \varepsilon)$ in (26). In all plots, we set $\gamma_2 = 10$, and $\varepsilon = 10^{-5}$. We neglect the third-order term $O_{\ln(n)/n}$. We note that when the SNRs are comparable and n is not too large, CCP is superior to SUPnoRS when the user with the largest SNR has a relatively low rate.

In the second column in Fig. 3 we show the optimal power and rate split vs R_2 . We observe the sharp transition in β that marks when CCP outperforms SUPnoRS. Improved channel conditions of the strong user decrease the α at which this transition occurs. As the SNRs become more dissimilar, the portion of the achievable rate region boundary attained by CCP decreases. The right column in Fig. 3 shows the optimal reliability allocation vs α . We observe that the $\epsilon_{1,1}$ term indicates that in the optimal allocation the strong user recovers the cloud center with a very high reliability across the rate region. More generally, we see that a relaxation of reliability for a user recovering their message is optimal as the rate demands of that user increase.

Fig. 3 demonstrates sets of channel conditions in which both $\beta = 1$ and $\beta = 0$ are required to achieve the largest achievable rate region. In contrast, Fig. 4 show channel conditions in which only $\beta = 1$ (left, effectively CCP alone) and $\beta = 0$ (right, effectively SUPnoRS alone) are required to achieve the largest achievable rate regions when the rate of each user is larger than $\ln(n)/n$.

In Fig. 5 we present a plot showing the coding scheme used to achieve the largest achievable regions across a set of channel conditions $\gamma_1, \gamma_2 \in [2, 50]$ for $\varepsilon = 10^{-1}$ and a blocklength $n = 100$. For each point in the plot, we evaluated the CCP, SUP, and SUPnoRS regions. The points are colored based on the “simplest” coding scheme that achieves the largest achievable region for meaningful rates, that is, rates larger than $\ln(n)/n$, for both users. Here “simplicity” is a somewhat arbitrary measure we define as $\{\text{CCP}, \text{SUPnoRS}, \text{SUP}\}$ with complexity increasing from left to right. This intuitively corresponds to the complexity of the coding scheme implementation by broadcaster and receiver, but more importantly we use it to illustrate the fact that for a very large set of channel conditions and reliability requirements, rate splitting (either as part of SUP or alone as CCP) is required to achieve the largest achievable rate regions. In the global reliability case this plot is symmetric about the line $\gamma_1 = \gamma_2$ so only the top half is plotted.

We now show plots for the per-user error requirements. In Fig. 6 we present $\mathcal{R}^{(\text{CCP})}(n, \epsilon_1, \epsilon_2)$, $\mathcal{R}^{(\text{SUP})}(n, \epsilon_1, \epsilon_2)$, and $\mathcal{R}^{(\text{CS})}(n, \epsilon_1, \epsilon_2)$ for four scenarios. In all scenarios the SNRs are $\gamma_1 = 35$ and $\gamma_2 = 30$. The scenario’s blocklength and reliability requirements are varied. For the top row $n = 100$, and for the bottom row $n = 5000$. The reliability constraints are varied from left to right. On the left, user 2 has a more relaxed reliability requirement of 0.9% and user 1 has a

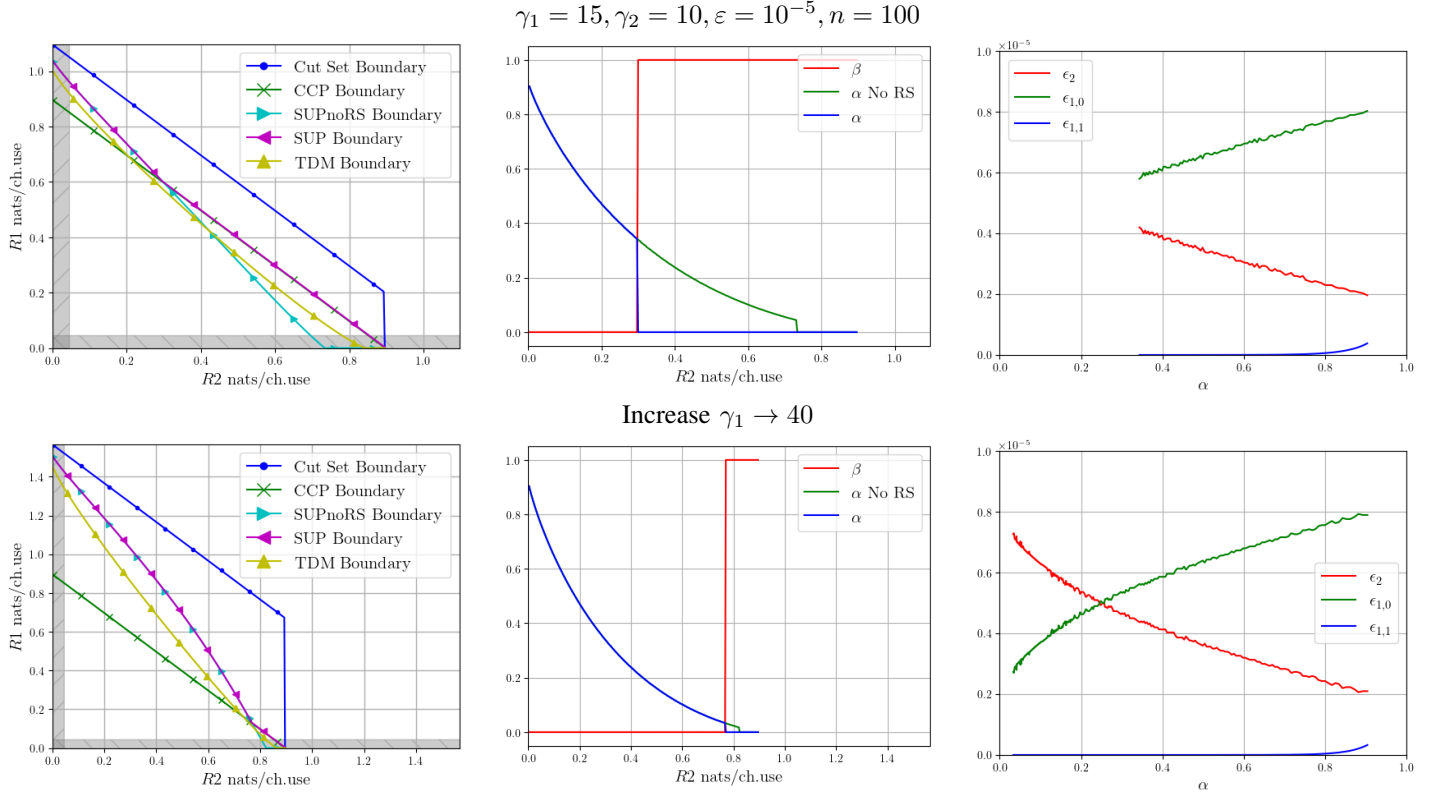


Fig. 3: Left: Achievable rate regions Center: Power and rate split allocations as a function of the rate allocated to user 2. Right: Reliability allocations as function of the optimal power split.

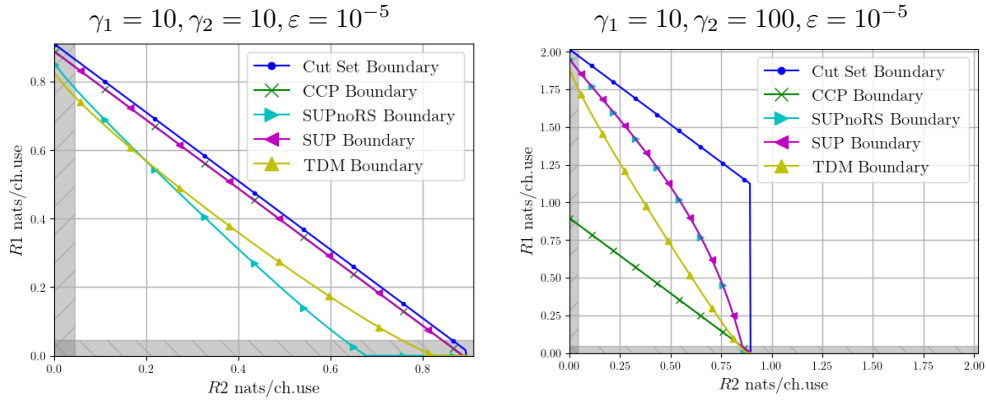


Fig. 4: Rate Regions showing (left) uniform out-performance of CCP, (right) uniform out-performance of SUPnoRS.

high reliability requirement of 99.999%. When user 2 has a larger point-to-point second order capacity (top left), a larger achievable rate region is found by encoding user 1's message in the cloud center. When n is increased to 5000, user 2 no longer has a larger point-to-point second order capacity and the capacity-achieving superposition ordering provides the largest achievable rate region. On the right, the plots maintain a similar shape as n is increased as the point-to-point second order capacity ordering does not change.

In Fig. 7 we present (as in Fig. 5) the coding schemes that achieve the largest achievable second order rate region for thousands of combinations of channel conditions. In each case

user 2 has a higher reliability requirement of 99.999% while the reliability of user 1 is 90%. The blocklength is fixed at $n = 100$. Points marked as SUP-1 are channel conditions and reliability requirements where encoding user 2's message in the cloud center gives the largest region. Points marked SUP-2 are channel conditions where encoding user 1's message in the cloud center gives the largest region. Points marked as CCP are channel conditions in which neither SUP ordering produces points on the achievable rate region boundary beyond what is produced by CCP. The achievable rate region formed by either SUP-1 or SUP-2 consists only of points where $\beta = 1$. This band clusters around and includes the line where the P2P

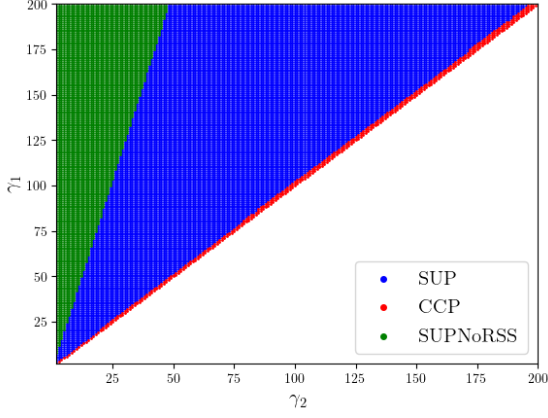


Fig. 5: ‘Simplest’ coding scheme required to obtain the largest achievable rate region for two users with varying channel conditions and a constant $\varepsilon = 0.1$ and $n = 100$.

second order capacities are equal. Finally, unmarked points correspond to channel conditions in which rate splitting is not required to achieve the largest region for any rate larger than $\ln(n)/n$. In these cases, a standard capacity achieving superposition code scheme achieves the best known finite blocklength achievable rate region.

V. CONVERSE BOUND PROOF

We shall set $R_0 = 0$ at the beginning of this section in order to simplify the notation. We shall also omit to explicitly write the event $\{(W_1, W_2) \text{ sent}\}$ within the probabilities of error. For the two-user AWGN BC with global error bounded by ε , we trivially have

$$1 - \varepsilon \leq \Pr[\text{dec}_1(\mathbf{Y}_1^n) = W_1 \cap \text{dec}_2(\mathbf{Y}_2^n) = W_2] \quad (41a)$$

$$\leq \min\{\Pr[\text{dec}_{\text{genie}}(\mathbf{Y}_1^n, \mathbf{Y}_2^n) = (W_1, W_2)], \quad (41b)$$

$$\Pr[\text{dec}_1(\mathbf{Y}_1^n) = W_1], \quad (41c)$$

$$\Pr[\text{dec}_2(\mathbf{Y}_2^n) = W_2]\}, \quad (41d)$$

where each of the terms in the minimum function in (41) relates to the performance of a Gaussian point-to-point channel. In particular, the probability of correct decoding in (41b) is that of a genie-aided receiver that has both channel outputs, and those in (41c) and (41d) correspond to considering the requirement for one of the users only. Therefore, an outer bound for $\mathcal{C}(n, \varepsilon)$ from (41) is

$$\mathcal{C}(n, \varepsilon) \subseteq \left\{ (R_1, R_2) \in \mathbb{R}_+^2 : \right. \quad (42a)$$

$$R_1 \leq \kappa(n, \gamma_1, \varepsilon) + O_{\ln(n)/n}, \quad (42b)$$

$$R_2 \leq \kappa(n, \gamma_2, \varepsilon) + O_{\ln(n)/n}, \quad (42c)$$

$$\left. R_1 + R_2 \leq \kappa_{\text{SIMO}}(n, \gamma_1, \gamma_2, \varepsilon) + O_{\ln(n)/n} \right\}, \quad (42d)$$

where $\kappa_{\text{SIMO}}(n, \gamma_1, \gamma_2, \varepsilon)$ in (42d) is the second order normal approximation for the Gaussian point-to-point SIMO channel (with SNRs at the two receive antennas given by γ_1 and γ_2) with error rate ε ; this bound depends on the correlation on the

noises on the two antennas. In [1] we wrote that the sum-rate in (42d) can be replaced by

$$R_1 + R_2 \leq \kappa(n, \max(\gamma_1, \gamma_2), \varepsilon) + O_{\ln(n)/n}, \quad (43)$$

which is true only for the physically degraded BC; in this case $Y_2 = Y_1 + Z_0$ with $Z_0 \sim \mathcal{N}(0, \sigma_2^2 - \sigma_1^2)$ independent of Z_1 , and thus $\text{dec}_{\text{genie}}(\mathbf{Y}_1^n, \mathbf{Y}_2^n) = \text{dec}(\mathbf{Y}_1^n)$, but for the general case we cannot draw the same conclusion. Next, we provide a derivation of [27, Corollary 1] that generalizes straightforwardly to any number of users. From the series of inclusions in (44) at the top of page 11, we can bound the sum-rate for the case of arbitrarily correlated noises as

$$R_1 + R_2 \leq \kappa(n, \max(\gamma_1, \gamma_2), 2\varepsilon) + O_{\ln(n)/n}. \quad (45)$$

Notice that the error term in (45) is 2ε , while in (43) it was ε . The sum-rate bound in (45) with the single-rate bounds in (42) proves the right hand side inclusion in Theorem 1, after including the common rate R_0 back in each bound.

A. Extension to K users

The reasoning in (44) extends to the case of K users and gives, in the case of private rates only, the bound

$$\mathcal{C}(n, \varepsilon) \subseteq \left\{ (R_1, R_2, \dots, R_K) \in \mathbb{R}_+^K : \forall S \subseteq [K] \right. \quad (46a)$$

$$\left. \sum_{j \in S} R_j \leq \kappa(n, \max\{\gamma_j : j \in S\}, |S|\varepsilon) + O_{\ln(n)/n} \right\}. \quad (46b)$$

With common rates, the sum “ $\sum_{j \in S} R_j$ ” in (46b) must be extended so as to include the rates of all the messages intended for the users indexed by the set S .

VI. ACHIEVABLE BOUND PROOF

Superposition coding with rate splitting and simultaneous joint decoding is capacity achieving for the more capable BC (and thus also for the stochastically degraded AWGN BC), and achieves [7, Sec 8.1]

$$\mathcal{C} = \bigcup_{(\alpha, \beta) \in [0, 1]^2} \left\{ (R_0, R_1, R_2) \in \mathbb{R}_+^3 : \right.$$

$$R_0 + R_2 + \beta R_1 \leq C \left(\frac{(1 - \alpha)\gamma_2}{1 + \alpha\gamma_2} \right), \quad (47a)$$

$$(1 - \beta)R_1 \leq C(\alpha\gamma_1), \quad (47b)$$

$$R_0 + R_1 + R_2 \leq C(\gamma_1) \left. \right\}, \quad (47c)$$

where α is the power split and β is the rate split. The constraint in (47c) is always redundant when $\gamma_1 \geq \gamma_2$, thus, the region in (47) is equivalent to (17), and $\beta = 0$ is always optimal. We aim to derive second order terms for (47).

a) Rate Splitting: The message for user 1 is split as $m_1 = (m_{10}, m_{11})$, $\forall m_1 \in [M_1]$, here $m_{1j} \in [M_{1j}]$, $j \in \{0, 1\}$ and $M_{10}M_{11} = M_1$. We only split the message of the strong user to mimic the scheme in [7, Problem 6.18].

We aim to construct a superposition coding scheme where the cloud center carries

$$m'_2 := (m_0, m_2, m_{10}), \quad (48)$$

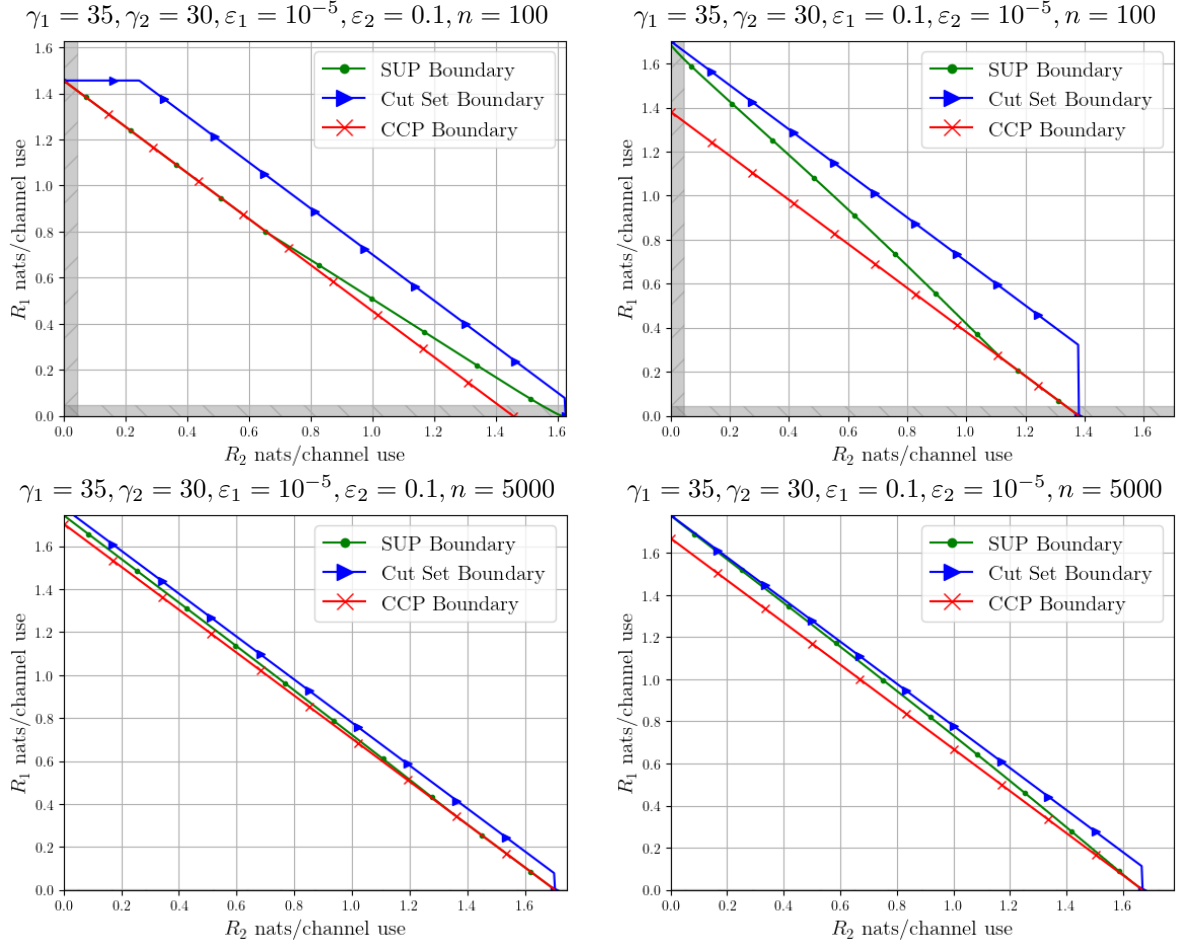


Fig. 6: Per-user error constraint achievable rate regions.

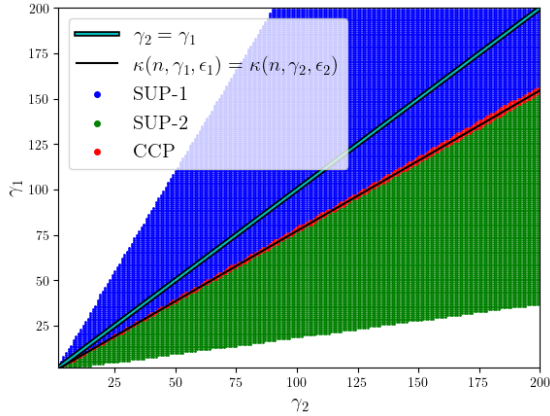


Fig. 7: ‘Simplest’ coding scheme required to obtain the largest achievable rate region for two users with varying channel conditions and a constant $\epsilon_1 = 0.1$, $\epsilon_2 = 0.001$ and $n = 100$.

and the satellite carries

$$m'_1 := m_{11}, \quad (49)$$

As for the decoding, receiver 2 decodes the cloud center only, while receiver 1 decodes both by SIM. Our code construction is inspired by [26] for the MAC with degraded message sets:

receiver 1 is exactly the same as the receiver in [26], but in addition we must consider the decoding constraint of receiver 2 that only decodes the cloud center while treating the satellite codeword as noise. In addition we also need to include the power constraint at the transmitter. The details of the scheme are presented next.

b) Random Code Construction on the Power Sphere:

For a power constraint $P > 0$, fix real numbers $(\rho, P_1, P_2) \in [-1, 1] \times \mathbb{R}_+ \times \mathbb{R}_+$ such that¹

$$(1 - \rho^2)P_1 + (\sqrt{P_2} + \rho\sqrt{P_1})^2 = P. \quad (50)$$

We further parameterize (50) as follows, for some $\alpha \in [0, 1]$,

$$(1 - \rho^2)P_1 = \alpha P, \quad (51a)$$

$$(\sqrt{P_2} + \rho\sqrt{P_1})^2 = \xi^2 P_2 = (1 - \alpha)P, \quad (51b)$$

$$\xi := 1 + \rho\sqrt{P_1/P_2}. \quad (51c)$$

¹ As our proof will show later on, it suffices to consider $\rho = 0$ in the following. This is so because geometrically [26] we can write the pair $(\mathbf{x}_1, \mathbf{x}_2)$ in $\mathcal{D}_n(\rho, P_1, P_2)$ in (52) as

$$\mathbf{x} = \mathbf{a}_2 + \mathbf{a}_1 : \begin{cases} \mathbf{a}_2 \in \mathcal{S}_{n-1}(\sqrt{n(1-\alpha)P}) \\ \mathbf{a}_1 \in \mathcal{S}_{n-2}(\sqrt{n\alpha P}) \\ \langle \mathbf{a}_1, \mathbf{a}_2 \rangle = 0 \end{cases}.$$

We decided to describe the scheme with any $\rho \in [-1, 1]$ to make the code construction, and thus its analysis, to be essentially the same as in [26].

Let $E_1 = \{\text{dec}_1(\mathbf{Y}_1^n) \neq W_1\}$ and $E_2 = \{\text{dec}_2(\mathbf{Y}_2^n) \neq W_2\}$ be the error events at the receivers. For $\gamma_1 \geq \gamma_2$ we have

$$\mathcal{C}(n, \varepsilon) = \cup_{\text{enc}, \text{dec}_1, \text{dec}_2} \{(R_1, R_2) : \Pr[E_1 \cup E_2] \leq \varepsilon\} \quad (44a)$$

$$= \cup_{\text{enc}, \text{dec}_1, \text{dec}_2} \{(R_1, R_2) : \Pr[E_1] + \Pr[E_2 \setminus E_1] + \Pr[E_2] + \Pr[E_1 \setminus E_2] \leq 2\varepsilon\} \quad (44b)$$

$$\subseteq \cup_{\text{enc}, \text{dec}_1, \text{dec}_2} \{(R_1, R_2) : \Pr[E_1] + \Pr[E_2] \leq 2\varepsilon\} \quad (44c)$$

$$= \cup_{\text{enc}, \text{dec}_1, \text{dec}_2} \{(R_1, R_2) : \Pr[\text{dec}_1(\mathbf{Y}_1^n) \neq W_1] + \Pr[\text{dec}_2(\mathbf{Y}_1^n + \mathbf{Z}_0^n) \neq W_2] \leq 2\varepsilon\} \quad (44d)$$

$$\subseteq \cup_{\text{enc}, \text{dec}_1, \text{dec}_2} \{(R_1, R_2) : \Pr[\text{dec}_1(\mathbf{Y}_1^n) \neq W_1] + \Pr[\text{dec}_2(\mathbf{Y}_1^n) \neq W_2] \leq 2\varepsilon\} \quad (44e)$$

$$\subseteq \cup_{\text{enc}, \text{dec}_0} \{(R_1, R_2) : \Pr[\text{dec}_0(\mathbf{Y}_1^n) \neq (W_1, W_2)] \leq 2\varepsilon\}, \quad (44f)$$

where in (44d) we used $\mathbf{Y}_2^n \sim \mathbf{Y}_1^n + \mathbf{Z}_0^n$, and in (44e) the monotonicity in SNR.

In order to write (51c) we implicitly assumed $P_2 > 0$, or equivalently $\alpha \neq 1$; the extreme cases $\alpha = 0$ and $\alpha = 1$ will be analyzed separately in the following. The codebook is composed of triplets $(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}) \in \mathbb{R}^{3n}$ from the set

$$\mathcal{D}_n(\rho, P_1, P_2) := \{(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}) \in \mathbb{R}^{3n} : \mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2, \quad (52a)$$

$$\|\mathbf{x}_1\|^2 = nP_1, \quad \|\mathbf{x}_2\|^2 = nP_2, \quad (52b)$$

$$\langle \mathbf{x}_1, \mathbf{x}_2 \rangle = n\rho\sqrt{P_1 P_2}\}. \quad (52c)$$

A transmitted codeword \mathbf{x} in (52) satisfies, because of (50),

$$\|\mathbf{x}\|^2 = \|\mathbf{x}_1\|^2 + \|\mathbf{x}_2\|^2 + 2\langle \mathbf{x}_1, \mathbf{x}_2 \rangle \quad (53a)$$

$$= nP_1 + nP_2 + 2n\rho\sqrt{P_1 P_2} = nP, \quad (53b)$$

i.e., codewords in $\mathcal{D}_n(\rho, P_1, P_2)$ meet the power constraint with equality. The codewords are chosen independently uniformly at random on their respective power sphere.

c) *Threshold Decoders*: The channel transition probabilities are $W_j(y|x) = \mathcal{N}(y; x, \sigma_j^2)$, $j \in [2]$. Let $P_{\mathbf{X}_2, \mathbf{X}}$ be the joint distribution induced by the codebook generation, namely

$$P_{\mathbf{X}_2, \mathbf{X}}(\mathbf{u}, \mathbf{x}) = P_{\mathbf{X}_2}(\mathbf{u})P_{\mathbf{X}|\mathbf{X}_2}(\mathbf{x}|\mathbf{u}) \quad (54a)$$

$$= \frac{\delta(\|\mathbf{u}\|^2 - nP_2)}{S_n(\sqrt{nP_2})} \quad (54b)$$

$$\cdot \frac{\delta(\|\mathbf{x} - \mathbf{u}\|^2 - nP_1, \langle \mathbf{x} - \mathbf{u}, \mathbf{u} \rangle - n\rho\sqrt{P_1 P_2})}{\sqrt{nP} S_{n-1}(\sqrt{n(1-\rho^2)P_1})}, \quad (54c)$$

where the function $S_n(\cdot)$ was defined in (2), which induces

$$P_{\mathbf{X}}(\mathbf{x}) = \int P_{\mathbf{X}_2, \mathbf{X}}(\mathbf{u}, \mathbf{x}) d\mathbf{u} = \frac{\delta(\|\mathbf{x}\|^2 - nP)}{S_n(\sqrt{nP})}. \quad (54d)$$

Thus for $j \in [2]$ we can compute

$$P_{\mathbf{Y}_j|\mathbf{X}_2}(\mathbf{y}|\mathbf{u}) = \int P_{\mathbf{X}|\mathbf{X}_2}(\mathbf{x}|\mathbf{u})W_j^n(\mathbf{y}|\mathbf{x}) d\mathbf{x}, \quad (54e)$$

$$P_{\mathbf{Y}_j}(\mathbf{y}) = \int P_{\mathbf{X}}(\mathbf{x})W_j^n(\mathbf{y}|\mathbf{x}) d\mathbf{x}, \quad (54f)$$

for $P_{\mathbf{X}|\mathbf{X}_2}$ in (54c) and $P_{\mathbf{X}}$ in (54d).

In the following we shall use the n -fold product of

$$Q_{X_2, X, Y_j}(u, x, y) = Q_{X_2}(u)Q_{X|X_2}(x|u)W_j(y|x) \quad (55a)$$

$$= \mathcal{N}\left(\begin{bmatrix} u \\ x \\ y \end{bmatrix}; \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} P_2 & P_2\xi & P_2\xi \\ P_2\xi & P & P \\ P_2\xi & P & P + \sigma_j^2 \end{bmatrix}\right), \quad (55b)$$

whose (conditional) marginals are

$$Q_{Y_j|X}(y|x) = \mathcal{N}(y; x, \sigma_j^2) = W_j(y|x), \quad (55c)$$

$$Q_{Y_j|X_2}(y|u) = \mathcal{N}(y; \xi u, (1 - \rho^2)P_1 + \sigma_j^2), \quad (55d)$$

$$Q_{Y_j}(y) = \mathcal{N}(y; 0, P + \sigma_j^2). \quad (55e)$$

From the rate split, let $M'_1 := M_{11}$ and $M'_2 := M_{10}M_0M_2$, therefore $M_0M_1M_2 = M'_1M'_2$, and

$$R'_{j,n} := \frac{1}{n} \ln(M'_j), \quad j \in [2]. \quad (56)$$

Also define the mutual information densities

$$i_{j,2}(\mathbf{y}; \mathbf{x}(m'_1, m'_2)) := \frac{1}{n} \ln \frac{W_j^n(\mathbf{y}|\mathbf{x}(m'_1, m'_2))}{Q_{Y_j}^n(\mathbf{y})}, \quad (57)$$

$$i_{j,1}(\mathbf{y}; \mathbf{x}(m'_1, m'_2)|\mathbf{x}_2(m'_2)) := \frac{1}{n} \ln \frac{W_j^n(\mathbf{y}|\mathbf{x}(m'_1, m'_2))}{Q_{Y_j|X_2}^n(\mathbf{y}|\mathbf{x}_2(m'_2))}, \quad (58)$$

$$i_{j,0}(\mathbf{y}; \mathbf{x}_2(m'_2)) := \frac{1}{n} \ln \frac{Q_{Y_j|X_2}^n(\mathbf{y}|\mathbf{x}_2(m'_2))}{Q_{Y_j}^n(\mathbf{y})} \quad (59)$$

$$= i_{j,2}(\mathbf{y}; \mathbf{x}(m'_1, m'_2)) - i_{j,1}(\mathbf{y}; \mathbf{x}(m'_1, m'_2)|\mathbf{x}_2(m'_2)),$$

where Q^n denotes the n -fold product of the distribution Q .

We employ threshold decoders. Receiver 1 looks for a unique pair $(m'_1, m'_2) \in [M'_1] \times [M'_2]$ that satisfies

$$\begin{cases} i_{1,2}(\mathbf{Y}_1; \mathbf{x}(m'_1, m'_2)) & > R'_{1,n} + R'_{2,n} + \gamma; \\ i_{1,1}(\mathbf{Y}_1; \mathbf{x}(m'_1, m'_2)|\mathbf{x}_2(m'_2)) & > R'_{1,n} + \gamma \end{cases}; \quad (60)$$

for some γ ; if none or more than one pair of indices are found in (60), receiver 1 declares an error. Receiver 2 looks for a unique $m'_2 \in [M'_2]$ that satisfies

$$i_{2,0}(\mathbf{Y}_2; \mathbf{x}_2(m'_2)) > R'_{2,n} + \gamma; \quad (61)$$

if none or more than one index is found in (61), receiver 2 declares an error.

d) *Performance Analysis for $\alpha \in (0, 1)$* : The average probability of error, averaged over the messages and over the random code construction, is bounded similarly to the standard typicality decoder [36] as

$$\epsilon_n \leq 1 - \Pr \left[\begin{cases} i_{2,0}(\mathbf{Y}_2; \mathbf{X}_2) > R'_{2,n} + \gamma \\ i_{1,1}(\mathbf{Y}_1; \mathbf{X}|\mathbf{X}_2) > R'_{1,n} + \gamma \\ i_{1,2}(\mathbf{Y}_1; \mathbf{X}) > R'_{1,n} + R'_{2,n} + \gamma \end{cases} \right]_{P_0} \quad (62a)$$

$$+ K_2 M'_2 \Pr [i_{2,0}(\mathbf{Y}_2; \mathbf{X}_2) > R'_{2,n} + \gamma]_{P_2} \quad (62b)$$

$$+ K_1 M'_1 M'_2 \Pr [i_{1,2}(\mathbf{Y}_1; \mathbf{X}) > R'_{1,n} + R'_{2,n} + \gamma]_{P_1} \quad (62c)$$

$$+ K_0 M'_1 \Pr [i_{1,1}(\mathbf{Y}_1; \mathbf{X}|\mathbf{X}_2) > R'_{1,n} + \gamma]_{P_3} \quad (62d)$$

Note that there is no “power constraint violation” probability in (62) because we picked the codewords from the set $\mathcal{D}_n(\rho, P_1, P_2)$ in (52) to satisfy the power constraint with equality. In particular we have:

• Eq(62d) relates to the event that the receiver 1 has decoded correctly the transmitted cloud center but not the satellite. The probability is computed from the distribution

$$P_3 := P_{\mathbf{X}_2}(\mathbf{u}) P_{\mathbf{X}|\mathbf{X}_2}(\mathbf{x}|\mathbf{u}) Q_{Y_1|\mathbf{X}_2}^n(\mathbf{y}_1|\mathbf{u}). \quad (63)$$

The factor

$$K_0 = 27 \sqrt{\frac{\pi}{8}} \frac{1 + 2\gamma_1}{\sqrt{1 + 4\gamma_1}}, \quad (64)$$

is the penalty for changing the measure from $P_{\mathbf{Y}_1|\mathbf{X}_2}$ to $Q_{Y_1|\mathbf{X}_2}^n$, as proven in Lemma 2. Overall, as proven in Lemma 1 in eq(103), we have

$$\text{eq}(62d) \leq K_0 e^{-n\gamma} \text{ for } \gamma = \frac{\ln(n)}{2n} \frac{K_0}{\sqrt{n}}. \quad (65)$$

• Eq(62c) relates to the event that receiver 1 has not decoded correctly the transmitted cloud center and the satellite. The probability is computed from the distribution

$$P_1 := P_{\mathbf{X}_2}(\mathbf{u}) P_{\mathbf{X}|\mathbf{X}_2}(\mathbf{x}|\mathbf{u}) Q_{Y_1}^n(\mathbf{y}). \quad (66)$$

The factor K_1 is the penalty for changing the measure from $P_{\mathbf{Y}_1}$ to $Q_{Y_1}^n$, as proven in Lemma 3 for $j = 1$. Overall, as proven in Lemma 1 in eq(104), we have

$$\text{eq}(62c) \leq K_1 e^{-n\gamma} \text{ for } \gamma = \frac{\ln(n)}{2n} \frac{K_1}{\sqrt{n}}. \quad (67)$$

• Eq(62b) relates to the event that receiver 2 has not decoded correctly the transmitted cloud center. The probability is computed from the distribution

$$P_2 := P_{\mathbf{X}_2}(\mathbf{u}) P_{\mathbf{X}|\mathbf{X}_2}(\mathbf{x}|\mathbf{u}) Q_{Y_2}^n(\mathbf{y}). \quad (68)$$

The factor K_2 is because we changed the measure from $P_{\mathbf{Y}_2}$ to $Q_{Y_2}^n$, as proven in Lemma 3 for $j = 2$. Overall, as proven in Lemma 1 in eq(105), we have

$$\text{eq}(62b) \leq K_2 e^{-n\gamma} \text{ for } \gamma = \frac{\ln(n)}{2n} \frac{K_2}{\sqrt{n}}. \quad (69)$$

• Eq(62a) relates to the event that the transmitted codeword does not pass the threshold decoder tests. The probability is computed from the distribution

$$P_0 := P_{\mathbf{X}_2}(\mathbf{u}) P_{\mathbf{X}|\mathbf{X}_2}(\mathbf{x}|\mathbf{u}) W_1^n(\mathbf{y}_1|\mathbf{x}) W_2^n(\mathbf{y}_2|\mathbf{x}), \quad (70)$$

since the noises are assumed to be independent. Overall, by the multi-dimensional Berry-Essen theorem [26, Theorem 11] with $\gamma = \ln(n)/2n$, we have that the probability on the RHS of (62a) can be upper bounded as proved in (71) at the top of the next page. In our derivation we used the first and second order moments of the information density vector

$$\mathbf{i} := \begin{bmatrix} i_{2,0}(\mathbf{Y}_2; \mathbf{x}_2(m'_2)) \\ i_{1,1}(\mathbf{Y}_1; \mathbf{x}(m'_1, m'_2)|\mathbf{x}_2(m'_2)) \\ i_{1,2}(\mathbf{Y}_1; \mathbf{x}(m'_1, m'_2)) \end{bmatrix}, \quad (72)$$

conditioned on a given codeword pair $(\mathbf{x}(m'_1, m'_2), \mathbf{x}_2(m'_2))$ chosen from $\mathcal{D}_n(\rho, P_1, P_2)$. In (72) we have sums of independent random variables of the following type, where $Y_{j,t}$ is the channel output at time $t \in [n]$ at receiver $j \in [2]$

$$\ln \frac{W_j(Y_{j,t}|\mathbf{x}_t)}{Q_{Y_j|\mathbf{X}_2}(Y_{j,t}|\mathbf{u}_t)} = C(\alpha\gamma_j) + \frac{\zeta_{j,t}^2 - N_{j,t}^2 \alpha\gamma_j}{2(1 + \alpha\gamma_j)} + \frac{\zeta_{j,t} N_{j,t}}{1 + \alpha\gamma_j}, \quad (73)$$

$$\ln \frac{W_j(Y_{j,t}|\mathbf{x}_t)}{Q_{Y_j}(Y_{j,t})} = C(\gamma_j) + \frac{\nu_{j,t}^2 - N_{j,t}^2 \gamma_j}{2(1 + \gamma_j)} + \frac{\nu_{j,t} N_{j,t}}{1 + \gamma_j}, \quad (74)$$

where we introduced the normalized quantities

$$N_{j,t} := \frac{Y_{j,t} - x_t}{\sigma_j} \sim \mathcal{N}(0, 1), \quad (75)$$

$$\zeta_{j,t} := \frac{x_t - \xi \mathbf{x}_2}{\sigma_j}, \quad \nu_{j,t} := \frac{x_t}{\sigma_j}. \quad (76)$$

By summing over $t \in [n]$ in (76) and with the shorthand notation $(\mathbf{x}, \mathbf{x}_2)$ for $(\mathbf{x}(m'_1, m'_2), \mathbf{x}_2(m'_2))$, we obtain that the means of the random variables in (72) are

$$\mathbb{E}[i_{j,2}(\mathbf{x} + \mathbf{Z}_j; \mathbf{x})] = C(\gamma_j) + \frac{\|\mathbf{x}\|^2/\sigma_j^2 - n\gamma_j}{2n(1 + \gamma_j)}, \quad (77a)$$

$$\mathbb{E}[i_{j,1}(\mathbf{x} + \mathbf{Z}_j; \mathbf{x}|\mathbf{x}_2)] \quad (77b)$$

$$= C(\alpha\gamma_j) + \frac{\|\mathbf{x} - \xi \mathbf{x}_2\|^2/\sigma_j^2 - n\alpha\gamma_j}{2n(1 + \alpha\gamma_j)}, \quad (77c)$$

$$\mathbb{E}[i_{j,0}(\mathbf{x} + \mathbf{Z}_j; \mathbf{x}_2)] = \text{eq}(77a) - \text{eq}(77c), \quad (77d)$$

and the (co)variances are

$$n \text{Var}[i_{j,2}(\mathbf{x} + \mathbf{Z}_j; \mathbf{x})] \quad (78a)$$

$$= \frac{1}{2} \left(\frac{\gamma_j}{1 + \gamma_j} \right)^2 + \frac{\|\mathbf{x}\|^2/\sigma_j^2}{n(1 + \gamma_j)^2}, \quad (78b)$$

$$n \text{Var}[i_{j,1}(\mathbf{x} + \mathbf{Z}_j; \mathbf{x}|\mathbf{x}_2)] \quad (78c)$$

$$= \frac{1}{2} \left(\frac{\alpha\gamma_j}{1 + \alpha\gamma_j} \right)^2 + \frac{\|\mathbf{x} - \xi \mathbf{x}_2\|^2/\sigma_j^2}{n(1 + \alpha\gamma_j)^2}, \quad (78d)$$

$$n \text{Cov}[i_{j,2}(\mathbf{x} + \mathbf{Z}_j; \mathbf{x}), i_{j,1}(\mathbf{x} + \mathbf{Z}_j; \mathbf{x}|\mathbf{x}_2)] \quad (78e)$$

$$= \frac{1}{2} \frac{\alpha\gamma_j}{1 + \alpha\gamma_j} \frac{\gamma_j}{1 + \gamma_j} + \frac{\langle \mathbf{x} - \xi \mathbf{x}_2, \mathbf{x} \rangle / \sigma_j^2}{n(1 + \alpha\gamma_j)(1 + \gamma_j)}, \quad (78f)$$

$$n \text{Var}[i_{j,0}(\mathbf{x} + \mathbf{Z}_j; \mathbf{x}_2)] = \text{eq}(78b) + \text{eq}(78d) - 2 \cdot \text{eq}(78f), \quad (78g)$$

$$n \text{Cov}[i_{1,\ell_1}(\cdots), i_{2,\ell_2}(\cdots)] = 0, \forall (\ell_1, \ell_2) \in [0 : 2]^3, \quad (78h)$$

where (78h) follows because the noises at the two receivers are assumed to be independent. Thus, the information density vector in (72) has mean $\mathbb{E}[\mathbf{i}] = \boldsymbol{\mu}(\alpha) + \boldsymbol{\mu}(\mathbf{x}_2, \mathbf{x})$ with

$$\boldsymbol{\mu}(\alpha) := \begin{bmatrix} C(\gamma_2) - C(\alpha\gamma_2) \\ C(\alpha\gamma_1) \\ C(\gamma_1) \end{bmatrix}, \quad (79a)$$

$$\boldsymbol{\mu}(\mathbf{x}_2, \mathbf{x}) := \begin{bmatrix} \frac{\|\mathbf{x}\|^2/\sigma_2^2 - n\gamma_2}{n2(1 + \gamma_2)} - \frac{\|\mathbf{x} - \xi \mathbf{x}_2\|^2/\sigma_2^2 - n\alpha\gamma_2}{n2(1 + \alpha\gamma_2)} \\ \frac{\|\mathbf{x} - \xi \mathbf{x}_2\|^2/\sigma_1^2 - n\alpha\gamma_1}{n2(1 + \alpha\gamma_1)} \\ \frac{\|\mathbf{x}\|^2/\sigma_1^2 - n\gamma_1}{n2(1 + \gamma_1)} \end{bmatrix}, \quad (79b)$$

We now upper bound the probability in (62a) for a fixed pair $(\mathbf{x}_2, \mathbf{x})$ as

$$\Pr \left[\begin{bmatrix} i_{2,0}(\mathbf{Y}; \mathbf{x}_2) \\ i_{1,1}(\mathbf{Y}; \mathbf{x}; \mathbf{x}_2) \\ i_{1,2}(\mathbf{Y}; \mathbf{x}) \end{bmatrix} - \boldsymbol{\mu}(\alpha) - \boldsymbol{\mu}(\mathbf{x}_2, \mathbf{x}) > \begin{bmatrix} R'_{2,n} + \gamma \\ R'_{1,n} + \gamma \\ R'_{1,n} + R'_{2,n} + \gamma \end{bmatrix} - \boldsymbol{\mu}(\alpha) - \boldsymbol{\mu}(\mathbf{x}_2, \mathbf{x}) \right] \quad (71a)$$

$$\geq \Pr \left[\mathbf{Z} > \sqrt{n} \begin{bmatrix} R'_{2,n} + \gamma \\ R'_{1,n} + \gamma \\ R'_{1,n} + R'_{2,n} + \gamma \end{bmatrix} - \sqrt{n} \boldsymbol{\mu}(\alpha) - \sqrt{n} \boldsymbol{\mu}(\mathbf{x}_2, \mathbf{x}) \right]_{\mathbf{Z} \sim \mathcal{N}(\mathbf{0}_3; \mathbf{V}(\alpha) + \mathbf{V}(\mathbf{x}_2, \mathbf{x}))} - \frac{B}{\sqrt{n}} \quad (71b)$$

$$= \Pr \left[\mathbf{Z} < -\sqrt{n} \begin{bmatrix} R'_{2,n} + \gamma \\ R'_{1,n} + \gamma \\ R'_{1,n} + R'_{2,n} + \gamma \end{bmatrix} + \sqrt{n} \boldsymbol{\mu}(\alpha) + \sqrt{n} \boldsymbol{\mu}(\mathbf{x}_2, \mathbf{x}) \right]_{\mathbf{Z} \sim \mathcal{N}(\mathbf{0}_3; \mathbf{V}(\alpha) + \mathbf{V}(\mathbf{x}_2, \mathbf{x}))} - \frac{B}{\sqrt{n}} \quad (71c)$$

$$\stackrel{\gamma = \frac{\ln(n)}{2\sqrt{n}}}{=} \Pr_{(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}) \in \mathcal{D}_n(\rho, P_1, P_2)} \left(\Psi \left(-\sqrt{n} \begin{bmatrix} R'_{2,n} \\ R'_{1,n} \\ R'_{1,n} + R'_{2,n} \end{bmatrix} + \sqrt{n} \boldsymbol{\mu}(\alpha) - \frac{\ln(n)}{2\sqrt{n}} \mathbf{1}; \mathbf{V}(\alpha) \right) - \frac{B}{\sqrt{n}} \right), \quad (71d)$$

where the vectors $\boldsymbol{\mu}(\alpha)$ and $\boldsymbol{\mu}(\mathbf{x}_2, \mathbf{x})$ are defined in (79a) and (79b), respectively; where the matrices $\mathbf{V}(\alpha)$ and $\mathbf{V}(\mathbf{x}_2, \mathbf{x})$ are defined in (80a) and (8), respectively; where in (71b) we used the multi-variate Berry-Essen theorem, with B a bounded constant that is specified in Lemma 4; and where the function $\Psi(\cdot, \cdot)$ was defined in (3). We note that any $(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}) \in \mathcal{D}_n(\rho, P_1, P_2)$ satisfies $\boldsymbol{\mu}(\mathbf{x}_2, \mathbf{x}) = \mathbf{0}_3$, and $\mathbf{V}(\mathbf{x}_2, \mathbf{x}) = \mathbf{0}_{3 \times 3}$

and covariance matrix $n\text{Cov}[\mathbf{z}] = \mathbf{V}(\alpha) + \mathbf{V}(\mathbf{x}_2, \mathbf{x})$ with

$$\mathbf{V}(\alpha) = \begin{bmatrix} \mathbf{V}_2(\alpha) & 0 \\ 0 & \mathbf{V}_1(\alpha) \end{bmatrix}, \quad (80a)$$

$$\mathbf{V}_2(\alpha) := [\mathbf{V}'(\alpha\gamma_2, \gamma_2)], \quad (80b)$$

$$\mathbf{V}_1(\alpha) := \begin{bmatrix} \mathbf{V}(\alpha\gamma_1, \alpha\gamma_1) & \mathbf{V}(\alpha\gamma_1, \gamma_1) \\ \mathbf{V}(\alpha\gamma_1, \gamma_1) & \mathbf{V}(\gamma_1, \gamma_1) \end{bmatrix}, \quad (80c)$$

for $\mathbf{V}'(\cdot, \cdot)$ and $\mathbf{V}(\cdot, \cdot)$ defined in (21a) and (7), respectively, and

$$\mathbf{V}(\mathbf{x}_2, \mathbf{x}) := \begin{bmatrix} v_{2,11} + v_{2,22} - 2v_{2,12} & 0 & 0 \\ 0 & v_{1,11} & v_{1,12} \\ 0 & v_{1,12} & v_{1,22} \end{bmatrix}, \quad (80d)$$

$$v_{j,11} := \frac{\|\mathbf{x} - \xi \mathbf{x}_2\|^2 / \sigma_j^2}{n(1 + \alpha\gamma_j)^2} - \frac{\alpha\gamma_j}{(1 + \alpha\gamma_j)^2}, \quad (80e)$$

$$v_{j,22} := \frac{\|\mathbf{x}\|^2 / \sigma_j^2}{n(1 + \gamma_j)^2} - \frac{\gamma_j}{(1 + \gamma_j)^2}, \quad (80f)$$

$$v_{j,12} := \frac{\langle \mathbf{x} - \xi \mathbf{x}_2, \mathbf{x} \rangle / \sigma_j^2}{n(1 + \alpha\gamma_j)(1 + \gamma_j)} - \frac{\alpha\gamma_j}{(1 + \alpha\gamma_j)(1 + \gamma_j)}. \quad (80g)$$

By construction, every codeword satisfies $\boldsymbol{\mu}(\mathbf{x}_2, \mathbf{x}) = \mathbf{0}_3$, and $\mathbf{V}(\mathbf{x}_2, \mathbf{x}) = \mathbf{0}_{3 \times 3}$.

• By combining everything together, we obtain the relationship in (81) at the top of the next page. The set $\mathbf{Q}_{\text{inv}}(\varepsilon; \mathbf{V}(\alpha))$ in (81c) for the block diagonal covariance matrix $\mathbf{V}(\alpha)$ in (80a) can be written as

$$\mathbf{Q}_{\text{inv}}(\varepsilon; \mathbf{V}(\alpha)) = \{ \mathbf{a} \in \mathbb{R}^3 : \Pr[\mathbf{Z} \leq -\mathbf{a}] \geq 1 - \varepsilon \} \quad (82a)$$

$$= \left\{ \mathbf{a} \in \mathbb{R}^3 : a_i = -\sqrt{[\mathbf{V}(\alpha)]_{ii}} \mathbf{Q}^{-1}(\varepsilon_i), i \in [3], \right. \quad (82b)$$

$$\left. \text{for } (\varepsilon_{10}, \varepsilon_{11}, \varepsilon_2) \in [0, 1]^3 \text{ that satisfy} \right\} \quad (82c)$$

$$1 - \varepsilon \leq \Pr[G_1 \leq \mathbf{Q}^{-1}(\varepsilon_2)]_{G_1 \sim \mathcal{N}(0,1)} \quad (82d)$$

$$\cdot \Pr[G_2 \leq \mathbf{Q}^{-1}(\varepsilon_{10}), G_3 \leq \mathbf{Q}^{-1}(\varepsilon_{11})]_{\substack{G_2 \\ G_3} \sim \mathcal{N}(\mathbf{0}, \begin{bmatrix} 1 & r \\ r & 1 \end{bmatrix})} \quad (82e)$$

$$= (1 - \varepsilon_2) \mathbf{F}(\varepsilon_{10}, \varepsilon_{11}; r), \quad (82f)$$

where $\mathbf{F}(\varepsilon_{10}, \varepsilon_{11}; r)$ was defined in (24). To complete the proof the R' values are replaced by their rate split equivalents as defined by (48) and (49) and Fourier Motzkin Elimination is performed. This proves the achievability of $\mathcal{R}^{(\text{SUP})}(n, \varepsilon)$ in (20) for $\alpha \in (0, 1)$.

e) *Performance Analysis for $\alpha = 0$* : Here the step in the above derivation where we used the multivariate Berry-Essen theorem does not hold because the 3×3 covariance matrix $\mathbf{V}(0)$ (from $\mathbf{V}(\alpha)$ in (80a) evaluated for $\alpha = 0$) has rank 2. In this case, our scheme reduces to a standard point-to-point codebook on the power sphere, that is $\|\mathbf{x}(m_0, m_1, m_2)\|^2 = nP$ for all $(m_0, m_1, m_2) \in [M_0] \times [M_1] \times [M_2]$, and where each receiver $j \in [2]$ looks for the triplet (m_0, m_1, m_2) that satisfies $i_{j,2}(\mathbf{y}_j; \mathbf{x}(m_0, m_1, m_2)) > R'_{1,n} + R'_{2,n} + \gamma$. The analysis proceeds as done for $\alpha \in (0, 1)$, except that the information density vector has dimension 2 rather than 3. The resulting region is as in (20) for the choice $\beta = 1, \alpha = 0, (1 - \varepsilon_{11})(1 - \varepsilon_2) = 1 - \varepsilon$ (here ε_{10} does not matter).

f) *Performance Analysis for $\alpha = 1$* : Here too the 3×3 covariance matrix $\mathbf{V}(1)$ (from $\mathbf{V}(\alpha)$ in (80a) evaluated for $\alpha = 1$) has rank 2. In this case, $R_0 = R_2 = 0$. Our scheme reduces to a standard point-to-point codebook on the power sphere, that is, $\|\mathbf{x}(m_1)\|^2 = nP$ for all $(m_1) \in [M_1]$, and where receiver 1 looks for an index m_1 that satisfies $i_{1,2}(\mathbf{y}_1; \mathbf{x}(m_1)) > R'_{1,n} + \gamma$. Receiver 2 does not do anything. The analysis proceeds as in the point-to-point case. The resulting region is as in (20) for the choice $\beta = 0, \alpha = 1, \varepsilon_{10} = \varepsilon$ (here ε_{11} and ε_2 do not matter).

A. Extension to K users

For simplicity, we only consider private rates and no splitting here. WLOG we assume $\gamma_1 \geq \gamma_2 \geq \dots \gamma_K > 0$.

The probability of error, averaged over the random code construction, can be bounded as

$$\epsilon_n \leq 1 - \Psi \left(-\sqrt{n} \begin{bmatrix} R'_{2,n} - C(\gamma_2) + C(\alpha\gamma_2) \\ R'_{1,n} - C(\alpha\gamma_1) \\ R'_{1,n} + R'_{2,n} - C(\gamma_1) \end{bmatrix} - \frac{\ln(n)}{2\sqrt{n}} \mathbf{1}; \mathbf{V}(\alpha) \right) + \frac{B + K_1 + K_2 + K_0}{\sqrt{n}} \stackrel{\text{to meet constraint in Definition 2}}{\leq} \epsilon, \quad (81a)$$

$$\iff \begin{bmatrix} R'_{2,n} - C(\gamma_2) + C(\alpha\gamma_2) \\ R'_{1,n} - C(\alpha\gamma_1) \\ R'_{1,n} + R'_{2,n} - C(\gamma_1) \end{bmatrix} + \frac{\ln(n)}{2n} \mathbf{1} \in \frac{1}{\sqrt{n}} \mathbf{Q}_{\text{inv}} \left(\epsilon - \frac{B + K_1 + K_2 + K_0}{\sqrt{n}}; \mathbf{V}(\alpha) \right) \quad (81b)$$

$$\iff \begin{bmatrix} R'_{2,n} \\ R'_{1,n} \\ R'_{1,n} + R'_{2,n} \end{bmatrix} \in \begin{bmatrix} C(\gamma_2) - C(\alpha\gamma_2) \\ C(\alpha\gamma_1) \\ C(\gamma_1) \end{bmatrix} + \frac{1}{\sqrt{n}} \mathbf{Q}_{\text{inv}}(\epsilon; \mathbf{V}(\alpha)) + O\left(\frac{\ln(n)}{n}\right), \quad (81c)$$

where for (81a) the function $\Psi(\cdot, \cdot)$ was defined in (3), for (81b) the function $\mathbf{Q}_{\text{inv}}(\cdot; \cdot)$ was defined in (5), the covariance matrix $\mathbf{V}(\alpha)$ was defined in (80a), and where (81c) follows from the continuity of $\mathbf{Q}_{\text{inv}}(\epsilon; \cdot)$ in ϵ proved similarly to [26, Lemma 5 proved in Appendix C].

a) *Capacity Region:* The capacity region \mathcal{C} of the K -user degraded BC $X \rightarrow Y_1 \rightarrow Y_2 \dots \rightarrow Y_{K-1} \rightarrow Y_K$ is attained by superposition coding, where the K levels of superposition satisfy the Markov chain

$$U_K \rightarrow U_{K-1} \rightarrow \dots \rightarrow U_1 \rightarrow X. \quad (83)$$

For the AWGN BC, we have

$$\mathcal{C} = \bigcup \left\{ (R_1, R_2, \dots, R_K) \in \mathbb{R}_+^K : \forall k \in [K] \right. \quad (84a)$$

$$\left. R_k \leq C\left(\gamma_k \sum_{\ell \in [k]} \alpha_\ell\right) - C\left(\gamma_k \sum_{\ell \in [k-1]} \alpha_\ell\right) \right\}, \quad (84b)$$

where the union in (84a) is over the “power splits”

$$(\alpha_1, \dots, \alpha_K) \in [0, 1]^K : \sum_{\ell \in [K]} \alpha_\ell = 1. \quad (84c)$$

The capacity region in (84) is attained, for example, by mutually independent $U_k \sim \mathcal{N}(0, \alpha_k P)$, $\forall k \in [K]$, and $X = \sum_{k \in [K]} U_k$ in (83) such that (84c) holds.

b) *First-Order Superposition Coding Region:* Consider a fixed $(\alpha_1, \dots, \alpha_K)$ as in (84c). In order not to clutter the notation next we omit to explicitly state the dependence on $(\alpha_1, \dots, \alpha_K)$ of various quantities, unless necessary or not clear from the context. For the purpose of developing a second order region, we write the capacity achieving superposition coding region with Gaussian input, where user $j \in [K]$ jointly decodes all the messages intended for the users indexed by $\{j, j+1, \dots, K\}$, as follows

$$\bigcap_{j \in [K]} \begin{cases} R_K + R_{K-1} + \dots + R_j & \leq I_{j,K} - I_{j,j-1} \\ R_{K-1} + \dots + R_j & \leq I_{j,K-1} - I_{j,j-1} \\ \vdots & \\ R_{j-1} + R_j & \leq I_{j,j+1} - I_{j,j-1} \\ R_j & \leq I_{j,j} - I_{j,j-1} \end{cases} \quad (85)$$

where $I_{j,\ell}$ is the mutual information at receiver $j \in [K]$ to decodes the messages indexed by $\{1, \dots, \ell\} : \ell \in [0 : K]$

after having removed the effect of the messages indexed by $\{\ell+1, \dots, K\}$, that is,

$$I_{j,\ell} := I(X; Y_j | U_{\ell+1}^K) = C\left(\gamma_j \sum_{k \in [\ell]} \alpha_k\right), \quad (86a)$$

with the convention that $I_{j,0} = 0$ and $U_{K+1}^K = \emptyset$, which satisfy

$$0 = I_{j,0} \leq I_{j,1} \leq I_{j,2} \dots \leq I_{j,K} = C(\gamma_j). \quad (86b)$$

We next we aim to find a second order region for (85).

c) *Random Codebook Generation:* For $(\alpha_1, \dots, \alpha_K)$ as in (84c), define

$$\mathcal{D}_n(\alpha_1, \dots, \alpha_K) := \left\{ (\mathbf{x}_1, \dots, \mathbf{x}_K, \mathbf{x}) \in \mathbb{R}^{(K+1)n} : \right. \quad (87a)$$

$$\left. \mathbf{x} = \sum_{k \in [K]} \mathbf{x}_k, \right. \quad (87b)$$

$$\left. \langle \mathbf{x}_j, \mathbf{x}_\ell \rangle = \delta(\ell - j) n \alpha_j P, \forall (j, \ell) \in [K]^2 \right\}. \quad (87c)$$

As in Footnote 1 for the two-user case, we choose the sub-codeword \mathbf{x}_k independently and uniformly at random on the power sphere $\mathcal{S}_{n-K+k}(\sqrt{n\alpha_k P})$ and mutually orthogonal. The resulting transmitted codeword in (87b) meets the power constraint nP with equality. This construction aims to mimic a choice of independent Gaussian U_1, \dots, U_K in (83).

d) *Threshold Decoding:* Define auxiliary distributions

$$Q_{j,\ell}(y|v_{\ell+1}, \dots, v_K) = \mathcal{N}\left(y; \sum_{i \in [\ell+1:K]} v_i, \sigma_j^2 + P \sum_{i \in [\ell]} \alpha_i\right), \quad (88)$$

for $(j, \ell) \in [K] \times [0 : K]$, with the convention $\sum_{i \in [0]} \alpha_i = 0$ and $\sum_{i \in [K+1:K]} v_i = 0$; with this we have

$$Q_{j,0}(y|v_1, \dots, v_K) = W_j\left(y \middle| \sum_{i \in [K]} v_i\right). \quad (89)$$

Let $\gamma = \ln(n)/2n$. Receiver $j \in [K]$, upon receiving \mathbf{y}_j , looks for a unique $(m_j, m_{j+1}, \dots, m_K) \in [M_j] \times [M_{j+1}] \times \dots \times [M_K]$ such that

$$R_j + \dots + R_\ell > i_{j,\ell}(\mathbf{y}_j) - i_{j,j-1}(\mathbf{y}_j) + \gamma, \forall \ell \in [j : K], \quad (90)$$

where—omitting message indices for readability—we defined

$$i_{j,\ell}(\mathbf{Y}_j) := \frac{1}{n} \ln \frac{W_j^n(\mathbf{Y}_j | \mathbf{x}^n)}{Q_{j,\ell}^n(\mathbf{Y}_j | \mathbf{x}_{\ell+1}^n, \dots, \mathbf{x}_K^n)} \Big|_{\mathbf{Y}_j = \mathbf{x} + \mathbf{Z}_j}. \quad (91)$$

e) *Performance Analysis*: Define

$$P_{j,\ell} := \gamma_j \sum_{i \in [\ell]} \alpha_i, \quad (92a)$$

which satisfy

$$0 = P_{j,0} \leq P_{j,1} \leq \dots \leq P_{j,K} = \gamma_j. \quad (92b)$$

The analysis proceeds as for the two-user case but with information density vectors of larger dimension. For receiver $j \in [K]$, consider the $(K - j + 2)$ -dimensional information density random vector

$$[i_{j,j-1}(\mathbf{Y}_j); i_{j,j}(\mathbf{Y}_j); \dots; i_{j,K}(\mathbf{Y}_j)], \quad (92c)$$

whose mean vector and covariance matrix conditioned on a transmitted codeword from $\mathcal{D}_n(\alpha_1, \dots, \alpha_K)$ —omitting to explicitly state the conditioning for readability—have entries

$$\mathbb{E}[i_{j,\ell}(\mathbf{Y}_j)] = C(P_{j,\ell}); \quad (92d)$$

$$n\text{Var}[i_{j,\ell}(\mathbf{Y}_j)] = V(P_{j,\ell}); \quad (92e)$$

$$n\text{Cov}[i_{j,\ell_1}(\mathbf{Y}_j), i_{j,\ell_2}(\mathbf{Y}_j)] = V(P_{j,\min\{\ell_1, \ell_2\}}, P_{j,\max\{\ell_1, \ell_2\}}), \quad (92f)$$

$$n\text{Cov}[i_{a,\ell_1}(\mathbf{Y}_a), i_{b,\ell_2}(\mathbf{Y}_b)] = 0, \quad a \neq b. \quad (92g)$$

Next, for receiver $j \in [K]$, from the means and covariances in (92), we evaluate the mean vector

$$\mathbb{E}[\mathbf{i}_j] = \boldsymbol{\mu}_j(\alpha_1, \dots, \alpha_K), \quad (93a)$$

and the covariance matrix

$$n\text{Cov}[\mathbf{i}_j] = \mathbf{V}_j(\alpha_1, \dots, \alpha_K), \quad (93b)$$

of the $(K - j + 1)$ -dimensional random vector

$$\mathbf{i}_j := [i_{j,j}(\mathbf{Y}_j) - i_{j,j-1}(\mathbf{Y}_j); \dots; i_{j,K}(\mathbf{Y}_j) - i_{j,j-1}(\mathbf{Y}_j)], \quad (93c)$$

whose entries satisfy

$$\mathbb{E}[i_{j,\ell}(\mathbf{Y}_j) - i_{j,j-1}(\mathbf{Y}_j)] = C(P_{j,\ell}) - C(P_{j,j-1}), \quad (93d)$$

and for $j - 1 < \min(\ell_1, \ell_2)$

$$n\text{Cov}[i_{j,\ell_1}(\mathbf{Y}_j) - i_{j,j-1}(\mathbf{Y}_j), i_{j,\ell_2}(\mathbf{Y}_j) - i_{j,j-1}(\mathbf{Y}_j)] \quad (93e)$$

$$= V(P_{j,\min\{\ell_1, \ell_2\}}, P_{j,\max\{\ell_1, \ell_2\}}) + V(P_{j,j-1}) \quad (93f)$$

$$- V(P_{j,j-1}, P_{j,\ell_1}) - V(P_{j,j-1}, P_{j,\ell_2}). \quad (93g)$$

Finally, for independent noises (i.e., block diagonal dispersion matrix), we obtain that the following second order region is achievable with block-length n and global reliability ε

$$\bigcup_{\substack{\sum_{j \in [K]} \alpha_j \leq 1 \\ \prod_{j \in [K]} (1 - \epsilon_j) \geq 1 - \varepsilon}} \bigcap_{j \in [K]} \left\{ (R_1, R_2, \dots, R_K) \in \mathbb{R}_+^K : \right. \quad (94a)$$

$$\begin{bmatrix} R_K + R_{K-1} + \dots + R_j \\ R_{K-1} + \dots + R_j \\ \vdots \\ R_{j-1} + R_j \\ R_j \end{bmatrix} \in \boldsymbol{\mu}_j(\alpha_1, \dots, \alpha_K) \quad (94b)$$

$$+ \frac{1}{\sqrt{n}} Q_{\text{inv}}(\epsilon_j; \mathbf{V}_j(\alpha_1, \dots, \alpha_K)) \Big\} + O_{\ln(n)/n} \mathbf{1}, \quad (94c)$$

where the constraint $\sum_{j \in [K]} \alpha_j \leq 1$ represents how power is allocated across private messages and $\prod_{j \in [K]} (1 - \epsilon_j) \geq 1 - \varepsilon$ how reliability is allocated across receivers.

Remark 10 (On Per-User Error). Without the optimization over $\prod_{j \in [K]} (1 - \epsilon_j) \geq 1 - \varepsilon$, the region in (94) is achievable with per-user average error probability bounded by ϵ_j for receiver $j \in [K]$. With per-user error, all $K!$ superposition coding ordering should be considered. \square

Remark 11 (On Rate-Splitting for the K -user Case). The challenge with rate splitting is that the complexity of the optimization problem and Fourier-Motzkin elimination explodes exponentially in the number of users K . To see this, let $W_{\mathcal{S}}$ denote the message intended for the receivers indexed by $\mathcal{S} \subseteq [K]$. In principle, any rate splitting can be obtained by augmenting each message $W_{\mathcal{S}}$ with the parts of the other messages that we require the users indexed by \mathcal{S} to decode. For the degraded BC, superposition coding is optimal and this is the strategy that we have investigated in our multi-user extension. Consider the case of $K = 3$ users, where the receivers are ordered according to the Markov chain $X \rightarrow Y_1 \rightarrow Y_2 \rightarrow Y_3$; capacity is achieved by superposition coding in the following order

$$U_3^n(W'_3) \rightarrow U_2^n(W'_2) \rightarrow U_1^n(W'_1) \rightarrow X^n, \quad (95)$$

$$W'_3 := (W_{\{1,2,3\}}, W_{\{1,3\}}, W_{\{2,3\}}, W_{\{3\}}) \quad (96)$$

$$W'_2 := (W_{\{1,2\}}, W_{\{2\}}) \quad (97)$$

$$W'_1 := (W_{\{1\}}). \quad (98)$$

This Markov chain structure of superposition coding suggests that we could rate splitting as follows

$$U_3^n(W'_{3,3}, W'_{2,3}, W'_{1,3}) \rightarrow U_2^n(W'_{2,2}, W'_{1,2}) \rightarrow U_1^n(W'_{1,1}) \rightarrow X^n, \quad (99)$$

$$W'_1 = (W'_{1,3}, W'_{1,2}, W'_{1,1}) \quad (100)$$

$$W'_2 = (W'_{2,3}, W'_{2,2}) \quad (101)$$

$$W'_3 = (W'_{3,3}) \quad (102)$$

but as one can see the problem quickly becomes very complex as K increases. How to “tame” the complexity of the superposition coding with rate splitting for general K is outside of the scope of this paper and we leave this for future work. Regardless of the difficulty, one can see that the projection of a K user achievable rate region onto a subset of two users only will coincide with the two user rate region as presented in this paper. That is, one can easily specify a K user channel in which rate splitting will be required to achieve the largest known achievable rate region. \square

VII. CONCLUSIONS

In this paper we provided achievable and converse second order rate regions for the SISO AWGN BC with both global and per-user reliability constraints. In addition, for the two-user case, rate splitting is shown to enlarge the achievable region for a large set of channel conditions. Surprisingly, rate splitting is only required to achieve CCP, that is, to have all information bits encoded into a single codeword. Extensions to the K -user case were discussed. We note that our construction utilizes codewords on the power shell, which achieves a lower dispersion than utilizing an i.i.d Gaussian codebook. The second order terms in our achievable and converse regions do not match. Tightening the converse bound and enlarging the achievable bound (by considering for example Marton's coding for the finite blocklength) are part of ongoing work.

APPENDIX

Lemma 1 (Han-type bounds). *Similarly to [37] we have*

$$\Pr \left[\frac{1}{n} \ln \frac{W_1^n(\mathbf{Y}_1|\mathbf{X})}{Q_{Y_1|X_2}^n(\mathbf{Y}_1|\mathbf{X}_2)} > \frac{1}{n} \ln(M'_1) + \gamma \right]_{P_{\mathbf{X}_2, \mathbf{X}} Q_{Y_1|X_2}^n} \quad (103a)$$

$$= \int_{(\mathbf{u}, \mathbf{x}, \mathbf{y}) : \frac{1}{n} \ln \frac{W_1^n(\mathbf{y}|\mathbf{x})}{Q_{Y_1|X_2}^n(\mathbf{y}|\mathbf{x})} > \frac{1}{n} \ln(M'_1) + \gamma} P_{\mathbf{X}_2, \mathbf{X}}(\mathbf{u}, \mathbf{x}) Q_{Y_1|X_2}^n(\mathbf{y}|\mathbf{u}) d\mathbf{u} d\mathbf{x} d\mathbf{y} \quad (103b)$$

$$\leq \int_{(\mathbf{u}, \mathbf{x}, \mathbf{y}) : \frac{W_1^n(\mathbf{y}|\mathbf{x})}{M'_1 e^{n\gamma}} > Q_{Y_1|X_2}^n(\mathbf{y}|\mathbf{u})} P_{\mathbf{X}_2, \mathbf{X}}(\mathbf{u}, \mathbf{x}) \frac{e^{-n\gamma}}{M'_1} W_1^n(\mathbf{y}|\mathbf{x}) d\mathbf{u} d\mathbf{x} d\mathbf{y} \quad (103c)$$

$$\leq \frac{e^{-n\gamma}}{M'_1}. \quad (103d)$$

Similarly

$$\Pr \left[\frac{1}{n} \ln \frac{W_1^n(\mathbf{Y}_1|\mathbf{X})}{Q_{Y_1}^n(\mathbf{Y}_1)} > \frac{1}{n} \ln(M'_1 M'_2) + \gamma \right]_{P_{\mathbf{X}_2, \mathbf{X}} Q_{Y_1}^n} \leq \frac{e^{-n\gamma}}{M'_1 M'_2}, \quad (104a)$$

and

$$\Pr \left[\frac{1}{n} \ln \frac{Q_{Y_2|X_2}^n(\mathbf{Y}_2|\mathbf{X}_2)}{Q_{Y_2}^n(\mathbf{Y}_2)} > \frac{1}{n} \ln(M'_2) + \gamma \right]_{P_{\mathbf{X}_2, \mathbf{X}} Q_{Y_2}^n} \leq \frac{e^{-n\gamma}}{M'_2}. \quad (105a)$$

Lemma 2 (Constant K_0).

$$\sup_{\mathbf{u} \in \mathbb{R}^n, \mathbf{y} \in \mathbb{R}^n} \frac{P_{Y_1|X_2}(\mathbf{y}|\mathbf{u})}{Q_{Y_1|X_2}^n(\mathbf{y}|\mathbf{u})} \leq 27 \sqrt{\frac{\pi e}{8}} \frac{1 + 2\gamma_1}{\sqrt{1 + 4\gamma_1}} =: K_0. \quad (106)$$

Proof of Lemma 2. Our proof is similar, and leverages the results of [20]. Codewords are chosen from the set $\mathcal{D}_n(\rho, P_1, P_2)$. By the spherical symmetry of the system and by a rotation of the coordinate axis, we can take WLOG

$$\mathbf{x}_2(m'_2) = (0^{n-1}, \sqrt{nP_2}). \quad (107)$$

By the code construction defining $\mathcal{D}_n(\rho, P_1, P_2)$, we have

$$\mathbf{x}_1(m'_1, m'_2) = (a^{n-1}(m'_1), \rho\sqrt{nP_1}), \quad (108)$$

with $a^{n-1}(m'_1)$ drawn uniformly at random from the power sphere $\mathcal{S}_{n-2}(\sqrt{n(1-\rho^2)P_1})$. The transmitted codeword is

$$\mathbf{x}_1(m'_1, m'_2) + \mathbf{x}_2(m'_2) = (a^{n-1}(m'_1), \xi\sqrt{nP_2}), \quad (109)$$

or equivalently,

$$\mathbf{x}(m'_1, m'_2) - \xi\mathbf{x}_2(m'_2) = (a^{n-1}(m'_1), 0). \quad (110)$$

Recall $(1-\rho^2)P_1 = \alpha P$ from (51). Therefore, from (109), we see we can decompose $P_{Y_j|X_2}(\cdot|(0^{n-1}, \sqrt{nP_2}))$ (obtained by averaging over the distribution of $a^{n-1}(m'_1)$) into the product of two distributions: (i) the first $n-1$ coordinates have the distribution induced by the uniform distribution on the power-sphere at the output of a point-to-point Gaussian channel with average SNR per channel use $\frac{n(1-\rho^2)P_1}{(n-1)\sigma_j^2} = \frac{n}{n-1}\alpha\gamma_j$; and (ii) the last coordinate is $\mathcal{N}(\xi\sqrt{nP_2}, \sigma_j^2)$. From (55d), the reference distribution $Q_{Y_1|X_2}^n(\cdot|(0^{n-1}, \sqrt{nP_2}))$ is jointly Gaussian with mean $\xi\mathbf{x}_2(m'_2)$ and covariance matrix $(\sigma_j^2 + \alpha P)\mathbf{I}_n$, that is, (i) the first $n-1$ coordinates are i.i.d. $\mathcal{N}(0, \sigma_j^2 + \alpha P)$, and (ii) the last coordinate is $\mathcal{N}(\xi\sqrt{nP_2}, \sigma_j^2 + \alpha P)$. Therefore,

$$\frac{P_{Y_j|X_2}(\mathbf{y} |(0^{n-1}, \sqrt{nP_2}))}{Q_{Y_j|X_2}^n(\mathbf{y} |(0^{n-1}, \sqrt{nP_2}))} \quad (111a)$$

$$\leq \frac{\mathcal{N}(y_n; \xi\sqrt{nP_2}, \sigma_j^2)}{\mathcal{N}(y_n; \xi\sqrt{nP_2}, \sigma_j^2 + \alpha P)} \quad (111b)$$

$$\cdot \frac{\mathcal{N}(y^{n-1}; 0^{n-1}, (\sigma_j^2 + \frac{n}{n-1}\alpha P)\mathbf{I}_{n-1})}{\mathcal{N}(y^{n-1}; 0^{n-1}, (\sigma_j^2 + \alpha P)\mathbf{I}_{n-1})} \quad (111c)$$

$$\cdot 27 \sqrt{\frac{\pi}{8}} \frac{1 + \xi_n}{\sqrt{1 + 2\xi_n}} \Big|_{\xi_n := \frac{n}{n-1}(1-\rho^2)\frac{P_1}{\sigma_j^2}} \quad (111d)$$

$$\leq 1 \cdot \sqrt{e} \cdot 27 \sqrt{\frac{\pi}{8}} \frac{1 + 2\gamma_j}{\sqrt{1 + 4\gamma_j}}, \quad (111e)$$

where (111b) is the contribution of the last coordinate, where (111d) is from [20, Eq. 104], and where (111c) is to account for the average SNR per channel use equal to $\frac{n}{n-1}\alpha\gamma_j$ on the first $n-1$ coordinates, as opposed to $\alpha\gamma_j$.

Lemma 3 (Constants K_j 's).

$$\sup_{\mathbf{y} \in \mathbb{R}^n} \frac{P_{Y_j}(\mathbf{y})}{Q_{Y_j}^n(\mathbf{y})} \leq 27 \sqrt{\frac{\pi}{8}} \frac{1 + \gamma_j}{\sqrt{1 + 2\gamma_j}} =: K_j, j \in [2]. \quad (112)$$

Proof of Lemma 3. Let $\gamma_j = P/\sigma_j^2$. In [20, Eq. 43] it was proved that that (112) holds for (a) P_{Y_j} is the distribution induced by the uniform distribution on $\mathcal{S}_{n-1}(\sqrt{nP})$ at the output of a point-to-point Gaussian channel with average noise power σ_j^2 , and (b) $Q_{Y_j}^n(\mathbf{y})$ is the i.i.d. Gaussian distribution with zero mean and variance $\sigma_j^2 + P = \sigma_j^2(1 + \gamma_j)$. In [26] it was shown that our superposition code construction induces the uniform distribution on $\mathcal{S}_{n-1}(\sqrt{nP})$, and thus (112) holds to our AWGN BC scenario as well.

Lemma 4. *The multivariate Berry-Essen [26, Theorem 11, for $d = 3$] states that for all convex, Borel measurable subsets of \mathbb{R}_d , we have that the constant B in (81) satisfies*

$$B \leq \frac{k_3 z}{\sqrt{n} (\lambda_{\min}(\mathbf{V}(\alpha) + \mathbf{V}(\mathbf{x}_2, \mathbf{x})))^{3/2}}; \quad (113)$$

$$k_d = 42d^{1/4} + 16 \text{ from [38] for } d = 3, \quad (114)$$

where $z := \frac{1}{n} \sum_{t \in [n]} \mathbb{E} [(\theta_t^T \theta_t)^{3/2}]$ for

$$\theta_t := \begin{bmatrix} \frac{(1-N_{2,t}^2)\gamma_2 + 2\frac{x_t}{\sigma_2} N_{2,t}}{2(1+\gamma_2)} - \frac{(1-N_{2,t}^2)\alpha\gamma_2 + 2\frac{x_t - \xi x_{2,t}}{\sigma_2} N_{2,t}}{2(1+\alpha\gamma_2)} \\ \frac{(1-N_{1,t}^2)\alpha\gamma_1 + 2\frac{x_t - \xi x_{2,t}}{\sigma_1} N_{1,t}}{2(1+\alpha\gamma_1)} \\ \frac{(1-N_{1,t}^2)\gamma_1 + 2\frac{x_t}{\sigma_1} N_{1,t}}{2(1+\gamma_1)} \end{bmatrix}; \quad (115)$$

Note, that while [38] only directly applies to $\mathcal{N}(0, \mathbf{I})$, methods similar to [19, Corollary 8] can be applied for general covariance matrix \mathbf{V} .

Proof of Lemma 4. For terms with N_1 in $\theta_t^T \theta_t$:

$$\begin{aligned} & (a_1 N^2 + b_1 N + c_1)^2 + (a_2 N^2 + b_2 N + c_2)^2 \\ & \leq \max \left(a_1^2 + a_2^2, \frac{b_1^2 + b_2^2}{4}, c_1^2 + c_2^2 \right) (|N| + 1)^4 \\ & =: f^2 (|N| + 1)^4. \end{aligned}$$

For terms with N_2 in $\theta_t^T \theta_t$:

$$\begin{aligned} & (a_3 N^2 + b_3 N + c_3)^2 \\ & \leq \max(|a_3|^2, (|b_3|/2)^2, |c_3|^2) (|N| + 1)^4 =: g^2 (|N| + 1)^4. \end{aligned}$$

Therefore

$$\begin{aligned} z &:= \frac{1}{n} \sum_{t \in [n]} \mathbb{E} [(\theta_t^T \theta_t)^{3/2}], \\ &\leq \frac{1}{n} \sum_{t \in [n]} \mathbb{E} \left[(f_t^2 (|N_{1,t}| + 1)^4 + g_t^2 (|N_{2,t}| + 1)^4)^{3/2} \right], \\ &\leq \frac{\sqrt{2}}{n} \sum_{t \in [n]} \mathbb{E} [|f_t|^3 (|N_{1,t}| + 1)^6 + |g_t|^3 (|N_{2,t}| + 1)^6], \\ &= \frac{(76 + 94\sqrt{2/\pi})\sqrt{2}}{n} \sum_{t \in [n]} (|f_t|^3 + |g_t|^3), \\ &\leq \frac{(76 + 94\sqrt{2/\pi})\sqrt{2}}{n} \sum_{t \in [n]} (|f_t|^2 + |g_t|^2)^{3/2}, \end{aligned}$$

since in general, for $0 < r < p$ we have $(\sum_i |x_i|^p)^{1/p} = \|\mathbf{x}\|_p \leq \|\mathbf{x}\|_r \leq d^{\frac{1}{r} - \frac{1}{p}} \|\mathbf{x}\|_p$, thus for $d = r = 2$ and $p = 3$

$$\sqrt{x^2 + y^2} \leq 2^{\frac{1}{2} - \frac{1}{3}} (|x|^3 + |y|^3)^{1/3}. \quad (116)$$

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